

SOLUTIONS TO PROBLEMS

CONTAINED IN

A GEOMETRICAL TREATISE ON CONIC SECTIONS.

BY THE

REV. W. H. DREW, M.A.

OF ST. JOHN'S COLLEGE, CAMBRIDGE;

SECOND MASTER OF BLACKHEATH PROPRIETARY SCHOOL.

Cambridge:

MACMILLAN AND CO.

AND 23, HENRIETTA STREET, COVENT GARDEN,

London.

1862.

Uitrupea Jukrinke 11
Auss. No. 21718. Date 4. 2. 1902

PREFACE.

THE present work is designed to be a continuation of, and is intended to be used by the learner in connexion with, my treatise on "Geometrical Conic Sections." The object is not so much to furnish a key, as to illustrate the application of the principles of Geometry to the solution of questions in Conic Sections. It has been too much the custom to despise the use of geometrical methods, and to reject them as uncertain except in the solution of very elementary problems. The present work will, I hope, show that Geometry is a more effective and practical instrument in the treatment of this subject than is generally supposed.

Another object that I have likewise kept in view is to supply a connecting link by which the student may be led from the simple process of following the steps of a demonstration, to devising for himself the solution of an original geometrical problem. Most persons engaged in teaching must have felt the want of some intervening step to bridge over this difficulty. With this purpose the Solutions are all based on one or other of the propositions in the former treatise, and the student is left to

PREFACE.

modify the construction and figure according to the particular question under consideration. An amount of care and attention will thus be rendered necessary, which cannot fail to insure the complete understanding of the principles of the subject, and which will make it impossible for the learner to acquire an apparent but treacherous knowledge, by the undue use of a retentive memory. Prob. 31 of the Parabola, Prob. 55 of the Ellipse, and Prob. 32 of the Hyperbola, afford good illustrations of the plan I have pursued, which really indicates the way in which the properties were themselves suggested. /

It has been found convenient to refer occasionally to a *new edition* of the "Conic Sections," which is now in course of publication. In this edition such slight modifications, as experience has shown to be needed, will be introduced. A considerable number of *unsolved* problems will be added, and the work will be brought completely up to the requirements of the present time.

I am indebted for much valuable help in the preparation of these "Solutions," to Mr. A. Freeman, of St. John's College, and to Mr. F. R. Drew, of Sidney College.

W. H. DREW.

CONIC SECTIONS



PROBLEMS ON THE PARABOLA.

1. DRAW BQ at right angles to AB ; then

$$AS \cdot SQ = BS^2 = 4 AS^2,$$

$$\therefore SQ = 4 AS, \text{ and } AQ = 5 AS.$$

2. The triangles PNG , GPK are equiangular, and have the side PG common,

$$\therefore \text{they are equal in all respects,}$$

$$\therefore PK = NG = 2 AS.$$

$$3. SP = SG = 2 NG = 4 AS = BC.$$

4. Produce PQ to meet the axis in T , and join AQ ; then,

$$ST \cdot SA = SQ^2 = 4 AS^2,$$

$$\therefore ST = 4 AS, \text{ and } SN = 2 AS,$$

$$\therefore SA : SQ :: SN : SP,$$

$$\therefore \angle SQA = \angle SPN, \text{ or } \angle QSB = \angle PSB.$$

5. Since PSZ is a right angle, and SY perpendicular to PZ ,

$$\therefore PY \cdot PZ = SP^2,$$

$$\text{also } PY \cdot YZ = SY^2 = SA \cdot SP.$$

$$6. AN \cdot NL = PN^2 = 4 AS \cdot AN,$$

$$\therefore NL = 4 AS.$$

7. Let the tangent at P meet the latus rectum in L ; then

$$ZY : YL :: XA : AS,$$

CONIC SECTIONS.

$$\begin{aligned}ZY &= YL, \\ \text{and } SY &\text{ is perpendicular to } ZL, \\ \therefore SZ &= SL.\end{aligned}$$

8. Since $AT = AN$, and the angles at A are right angles,
 $\therefore AY = NY$.

Also, since the angles at N and Y are right angles, a circle may be described about $SNPY$,

$$\therefore TP \cdot TY = TN \cdot TS$$

9. Let the tangent at P meet the tangent at A in Y ; then since SYP is a right angle, the circle described about SNP will pass through Y .

$$\text{Also, since } AY^2 = AS \cdot AT = AS \cdot AN,$$

$$\therefore AY \text{ a tangent to the circle.}$$

$$\text{Now } AT : PN :: AT : TN,$$

$$\therefore AY = \frac{1}{2} PN.$$

10. Let the tangent, ordinate, and normal at P' meet the axis in T' , N' and G' ; then

$$\text{since } AN' = AT' \text{ and } N'N = NG',$$

$$\therefore AN = \frac{1}{2} T'G'.$$

$$\text{Now } PN^2 = 2 AS \cdot 2 AN = N'G' \cdot T'G' = P'G'^2,$$

$$\therefore PN = P'G'.$$

11. From any point R on the tangent at P draw the tangent RQ ; then the triangles SRQ , SPR are similar,

$$\therefore \text{the angle } SRQ = \text{the angle } SPR,$$

$$\therefore \text{the } \angle SRQ \text{ is independent of the point } R.$$

12. Draw NQ a tangent to the circle.

$$\text{Now } RN^2 : A'B^2 :: AN^2 : AA'^2,$$

$$\text{and } PN^2 : A'B^2 :: AN : AA',$$

$$:: AN \cdot AA' : AA'^2,$$

$$\therefore RN^2 - PN^2 : A'B^2 :: AN \cdot A'N : AA'^2,$$

$$\text{or } RP \cdot RP' : NQ^2 :: A'B^2 : AA'^2.$$

13. Let B be the point where the axis of the parabola is to touch the circle, and C the other given point.

Draw the tangents TC , TB intersecting in T ; also draw CN perpendicular to TG , and CG passing through the centre O of the circle; then

NT and NG will be respectively the subtangent and subnormal of the parabola required.

Bisect NT in A , and TG in S ; then A and S will be respectively the vertex and focus of the parabola.

14. Let QT be the tangent at the point Q .

Through Q draw the chord QOQ' ; and let WRO be drawn parallel to the axis meeting QT in W , the parabola in R , and the chord QOQ' in O .

Bisect QQ' in V , and draw the diameter TPV parallel to the axis, and join SP .

Now from similar triangles

$$QO : OW :: QV : VT,$$

$$:: QV^2 : QV \cdot VT,$$

$$:: 4SP \cdot PV : 2QV \cdot PV,$$

since $QV^2 = 4SP \cdot PV$ (*Prop. XV.*)

and $VT = 2PV$ (*Prop. XIV.*)

$$\therefore QO : OW :: 4SP : 2QV,$$

$$:: 4SP : QQ',$$

$$\therefore QO \cdot QQ' = 4SP \cdot OW.$$

But $QO \cdot OQ' = 4SP \cdot RO$ (*Prop. XVII.*)

$$\therefore QQ' : OQ' :: OW : RO,$$

$$\therefore QO : OQ' :: WR : RO.$$

15. Join SP ; then

$$PV = ST = SP = PO.$$

16. Draw VM at right angles to the axis; then

$$\text{since } VM : MS :: PN : NT,$$

$$\therefore MS = NT.$$

$$\text{But } PN^2 = 2 AS \cdot NT,$$

$$\therefore VM^2 = 2 AS \cdot SM,$$

\therefore the locus of V is a parabola, whose vertex is S , and latus rectum half that of the given parabola.

17. Produce QV , UK to meet the tangent at P in W and X ; then as in *Prop. XIX.* it can be proved that WV and XU are each equal to $4 SP$,

$\therefore WV$ and XU are equal and parallel,

$\therefore UV$ is parallel to the tangent at P .

18. By similar triangles

$$CB : CA :: AN : PN,$$

$$\therefore PN \cdot CR = AC \cdot AN,$$

$$\text{or } PN^2 = AC \cdot AN,$$

\therefore the locus of P is a parabola whose axis is AB , and latus rectum equal to the given distance AC .

19. Let CP and AR intersect in V ; and draw $VN \perp$ to AC , and $VM \perp$ to the tangent at A ; then

$$VN : AN :: CR : CA,$$

$$:: CR : CP,$$

$$:: VN : CV,$$

$$\therefore CV = AN = VM,$$

\therefore the locus of V is a parabola whose focus is C , and directrix AM .

20. Let OQ , OQ' be two equal tangents drawn from the point O in the axis; and let them be cut by the tangent RPR' in R and R' .

Join SP ; SQ , SQ' ; SR , SR' ; then

the $\angle SOR' =$ the $\angle SOQ =$ the $\angle SQO$;

and the $\angle SR'O =$ the supplement of $SR'Q$,

$$= \angle SPR' \text{ (Prop. XII.)}$$

$$= \text{the angle } SPR,$$

$$= \text{the angle } SRQ \text{ (Prop. XII.)}$$

\therefore in the triangles SQR' , SQR ,

the angles $\angle SOR'$, $\angle SR'O = \angle SQR$, $\angle SRQ$ each to each,

$$\therefore SO = SQ,$$

$$\therefore RQ = OR',$$

$$\text{and } OQ = OQ',$$

$$\therefore OR = R'Q'.$$

21. Join QE , $Q'E$; then

$$\text{the } \angle QEQ' = 2 \angle QQ'Q'.$$

Again, let OQ and OQ' meet the axis ASx in T and T' ; then

$$\angle QSx = \angle SQT + \angle STQ,$$

$$= 2 \angle STQ = 2 \angle QOV;$$

$$\text{so the } \angle Q'Q'Sx = 2 \angle ST'Q' = 2 \angle Q'OQ',$$

$$\therefore \text{the } \angle QSQ' = 2 \angle QOQ',$$

$$\therefore \text{the } \angle QSQ' = \text{the } \angle QEQ',$$

\therefore the circle described about the triangle QEQ' will pass through S .

If $Q'Q$ pass on the left of S , a *similar* proof will hold good.

It must then be shown that the angles QSQ' , QEQ' are together equal to two right angles.

22. Draw the ordinates PN , pn ; then

$$SN : Sn :: ST : Sp,$$

$$\therefore XN : Xn,$$

$$\begin{aligned}
&\therefore AN : XN :: Sn : Xn, \\
&\therefore XN + SN : XN - SN :: Xn + Sn : Xn - Sn, \\
&\therefore 2AN : 2AS :: 2AS : 2An, \\
&\quad \text{or } AN : AS :: AS : An, \\
&\therefore 4AS \cdot AN : 4AS^2 :: 4AS^2 : 4AS \cdot An; \\
&\quad \text{or } PN^2 : 4AS^2 :: 4AS^2 : pn^2, \\
&\therefore PN : 2AS :: 2AS : pn, \\
&\quad \text{or } PN : 4AS :: AS : pn.
\end{aligned}$$

Again,

$$\begin{aligned}
SQ : AS &:: PN : AN, \\
&:: 4AS \cdot PN : 4AS \cdot AN, \\
&:: 4AS \cdot PN : PN^2, \\
&:: 4AS : PN, \\
&:: pn : AS, \\
&\therefore SQ = pn, \\
&\text{so } Sq = PN.
\end{aligned}$$

23. Join QS and produce it to q ; then

$$\begin{aligned}
&\left. \begin{array}{l} SR \text{ bisects the angle } PSq, \\ \text{and } Sr \dots \dots \dots pSq, \end{array} \right\} (Prop. IV.) \\
&\therefore \text{the angle } RSr \text{ is half of the two } PSq \text{ and } pSq, \\
&\therefore \text{the angle } RSr \text{ is a right angle,} \\
&\therefore DR \cdot Dr = SD^2 = 4AS^2.
\end{aligned}$$

24. Join PQ , and draw OV parallel to the axis; then

$$\text{since } OQ' : OQ :: PV : QV,$$

$$\therefore Q'P \text{ is parallel to } OV,$$

$$\left\{ \begin{array}{l} \therefore \text{the } \angle OQ'P = \text{the } \angle QQV, \\ \quad = \text{the } \angle QTS, \text{ if } OQ \text{ meet the axis in } T, \\ \quad = \text{the } \angle SQO, \end{array} \right.$$

and the $\angle OPQ' = \text{the } \angle OPS,$
 $= \text{the } \angle SOQ \text{ (Prop. XII.)}$

PROBLEMS.

\therefore the triangle POQ is similar to OSQ ,

$$\therefore PQ : OP :: OQ : OS,$$

$$\therefore OS \cdot PQ = OQ \cdot OP.$$

25. Draw QV parallel to the tangent at P ; and draw SY perpendicular to the tangent, and join AY ; then

$$QD : QV :: SY : ST,$$

$$:: SY : SP,$$

$$\therefore QD^2 : QV^2 :: SY^2 : SP^2,$$

$$:: SP \cdot SA : SP^2,$$

$$:: SA : SP,$$

$$:: 4AS : 4SP,$$

$$:: 4AS \cdot PV : 4SP \cdot PV,$$

$$\therefore QD^2 = 4AS \cdot PV.$$

26. Draw the tangent PT at the point P meeting the axis in T ; then by Prob. 14

$$TA : AO :: PO : OQ.$$

$$\text{But } AT = AM,$$

$$\therefore AM : AO :: PO : OQ;$$

$$\text{So } AN : AO :: OQ : OP,$$

$$\therefore AM : AO :: AO : AN,$$

$$\therefore AM \cdot AN = AO^2.$$

27. By similar triangles

$$AM : PM :: QN : AN,$$

$$\therefore AM^2 : PM^2 :: QN^2 : AN^2,$$

$$\text{or } AM^2 : 4AS \cdot AM :: 4AS \cdot AN : AN^2,$$

$$\text{or } AM : 4AS :: 4AS : AN,$$

$$\therefore AM \cdot AN = (4AS)^2.$$

28. Let Q be the centre of the circle inscribed in the sector OAP .

Draw OQR , meeting the circles at their point of contact R .

Complete the quadrant AOB , and draw BN parallel to OA ; and through Q draw MQN , meeting OA and BN at right angles in M and N ; then

$$\begin{aligned} QR &= QM, \\ \text{and } OR &= MN, \\ \therefore OQ &= QN, \end{aligned}$$

\therefore the locus of Q is a parabola, whose focus is O , and directrix BN .

$$29. (1.) AN : AM :: PN^2 : QM^2$$

$$:: AN^2 : SM^2;$$

$$\text{or } AN^2 : AN \cdot AM :: AN^2 : SM^2,$$

$$\therefore SM^2 = AN \cdot AM.$$

(2.) Bisect QQ' in V , and draw VR parallel to the axis bisecting AP in R ; then

$$AS + AM = SQ,$$

$$AS + AM' = SQ',$$

$$\begin{aligned} \therefore MM' &= SQ - SQ', \\ &= 2SV = 2AR, \\ &= AP. \end{aligned}$$

$$30. \quad RV : Q'V' :: PV : PV',$$

$$:: QV^2 : Q'V'^2,$$

$$RV \cdot Q'V' : Q'V'^2 :: QV^2 : Q'V'^2,$$

$$\therefore RV \cdot Q'V' = QV^2;$$

$$\text{So } R'V' \cdot QV = Q'V'^2,$$

$$\therefore RV : R'V' :: QV^2 : Q'V'^2.$$

31. In the fig. of Prop. XVII. let Qq be the diameter of the circle, and $Q'q'$ the chord joining the two other points of intersection of the circle and parabola.

Join VV' meeting the axis in E ; VV' is at right angles to $Q'q$.

Now since the rectangle $QO \cdot Oq = Q'O \cdot Oq$,

$\therefore PV$ and PV' are equally distant from the axis,

$$\therefore VE = V'E.$$

Let $Qq, Q'q'$ meet the axis in F and F' ; and draw $VM, V'M'$ at right angles to the axis; then

$$FF' = MM' = 2EM = 2 \text{ subnormal of } P = 4AS.$$

32. Draw OPV parallel to the axis bisecting $Q'q$ in V ; then since the tangent at P is parallel to $Q'q$, it will be at right angles to OQ , and will therefore meet OQ on the directrix. Let this point be Z ; then

$$OZ : ZQ :: OP : PV,$$

$$\therefore OZ = ZQ.$$

33. The parabolas will intersect at the extremities B, C of the latus rectum. Draw the normal BG ; then the angle at which the parabolas intersect is twice the angle SBG , which is a right angle, since $SG = 2AS = SB$.

34. Let BSB' be the chord of the circle of curvature at B through the focus. Bisect BB' in C , and draw CO at right angles to BB' , meeting the diameter of curvature in O ; then since CO is parallel to the axis,

$$BO : BG :: BC : BS,$$

$$\text{and } BC = 2BS, \therefore BO = 2BG.$$

35. Draw PW parallel to the axis; then

$$\text{the angle } SPF = WPH = SHP,$$

and the angle at S is common to the triangles SPF, SHP ;

\therefore these triangles are similar,

$$\therefore SF : SP :: SP : SH,$$

$$\therefore SF \cdot SH = SP^2 = SG^2.$$

26. Let $\triangle PQQ'$ be the inscribed triangle.

Draw the tangents $QO, Q'O$, meeting in O ; and the tangent tPt' meeting OQ, OQ' in t and t' .

Produce $Q'P, QP$ to meet OQ, OQ' in q and q' .

Join $q'q$; then what we have to prove is that qq' , and the tangent at P will meet QQ' produced in the same point.

Draw $OV, PM, qm, q'm; tn, t'n'$, parallel to the axis of the parabola, meeting QQ' in V, M, m, m', n, n' . Produce MP to meet OQ in R , and let $qm, q'm'$ meet tPt' in r and r' .

The problem will clearly be solved if we can show that

$$qr : qm :: q'r' : q'm'.$$

Now

$$qm : Qm :: RM : QM,$$

$$Q'm : qm :: Q'M : PM,$$

$$\therefore Q'm : Qm :: RM \cdot Q'M : QM \cdot PM;$$

$$\text{but } RM : PM :: QQ' : Q'M \text{ (Prob. 14.)}$$

$$\therefore Q'm : Qm :: QQ' \cdot Q'M : QM \cdot Q'M,$$

$$:: QQ' : QM,$$

$$\therefore Qm : QQ' :: QM : QQ' + QM.$$

Again,

$$qr : PR :: rt : Pt,$$

$$:: 2mn : 2Mn,$$

$$:: 2Qm - 2Qn : 2Mn,$$

$$:: 2Qm - QM : QM \text{ (since } tn \text{ bisects } QM);$$

$$\text{but } 2Qm : QM :: 2QQ' : QQ' + QM,$$

$$\therefore 2Qm - QM : QM :: QQ' - QM : QQ' + QM,$$

$$\therefore qr : PR :: QQ' - QM : QQ' + QM,$$

$$\text{and } PR : PM :: QM : Q'M,$$

$$\therefore qr : PM :: QM : QQ' + QM,$$

$$:: Qm : QQ',$$

$$\therefore qr : Qm :: PM : QQ'.$$

In the same manner it can be shown that

$$q'r' : Q'm' :: PM : QQ',$$

$$\therefore qr : q'r' :: Qm : Q'm'.$$

$$\begin{aligned}
 \text{But} \quad Qm : qm &:: QV : OV, \\
 &:: Q'm' : q'm', \\
 \therefore Qm : Q'm' &:: qm : q'm', \\
 \therefore q'r : q'r' &:: qm : q'm', \\
 \text{or } qr : qm &:: q'r' : q'm'.
 \end{aligned}$$

37. Let the tangent RR' meet the curve in P ; and draw $Rm, PM, R'm'$ parallel to the axis meeting QQ' in m, M and m' .

$$\begin{aligned}
 \text{Now} \quad OR : RQ &:: Vm : Qm, \\
 &:: 2Vm : 2Qm, \\
 &:: 2QV - 2Qm : 2Qm, \\
 &:: QQ' - QM : QM, \\
 &:: QM : QM.
 \end{aligned}$$

$$\begin{aligned}
 \text{Similarly} \quad OR' : R'Q' &:: QM : Q'M, \\
 \therefore OR : RQ &:: R'Q' : Q'R'.
 \end{aligned}$$

38. Let AQ and AQ' be the chords, and R the further angle of the rectangle.

Join QQ' , and draw AR bisecting it in V . Also draw VP parallel to the axis, and at P draw the tangent PT parallel to QQ' .

Draw PN, VM, RL at right angles to the axis.

Make $A'O' = 2AO = 8AS$ (*Probs. 26 and 27*);

Then $RL = 2VM = 2PN$,

$$\begin{aligned}
 \text{and } O'L &= AL - A'O' = 2AM - 2AO, \\
 &= 2OM = 2TN = 4AN.
 \end{aligned}$$

But $PN^2 = 4AS \cdot AN$,

$$\therefore 4PN^2 = 4AS \cdot 4AN.$$

$$\text{or } RL^2 = 4AS \cdot O'L,$$

\therefore the locus of R is a parabola equal to the original one with its vertex at O' .

39. Draw AH at right angles to the tangent at P ; then if PK be the chord of curvature through A ,

$$PK : PU :: AH : AP \text{ (See fig. Prop. XIX.)}$$

$$\therefore PK \cdot AP = PU \cdot AH.$$

Again, $PU : 4SP :: SP : SY$ (Prop. XIX.)

$$:: TY : AY,$$

$$\therefore PU : PY :: 4SP : AY \text{ (since } PY = TY \text{).}$$

$$\text{Also, } AH : PY :: AY : SP,$$

$$\therefore PU \cdot AH : PY^2 :: 4SP \cdot AY : SP \cdot AY,$$

$$\therefore PU \cdot AH = 4PY^2.$$

$$\therefore PK \cdot AP = 4PY^2.$$

40. This problem is incorrect.

41. Let QQ' be a chord. Bisect it in V , and draw QM , VK , $Q'M'$ at right angles to the directrix; then

$$2VK = QM + Q'M'$$

$$= SQ + SQ'.$$

But $SQ + SQ' > QQ'$, unless QQ' pass through S ,

$$\therefore VK > QV.$$

Hence the circle described upon QQ' as diameter will only reach the directrix when the chord QQ' passes through the focus, in which case it will touch the directrix, since the angle VKM is a right angle.

42. Let A and A' be the vertices of the parabolas.

Produce the axes to meet in T , and draw the common tangent TP .

Join SS' , cutting PT at right angles in Y .

Join AY , and draw $S'X$ perpendicular to ST .

Now since AY is perpendicular to SX ,

$$\therefore SY : S'Y :: SA : AX.$$

But

$$SY = S'Y,$$

$$\therefore SA = AX,$$

$\therefore S'$ is on the directrix.

43. In the fig. of *Prop. XIX.* let Q' be the other point where the circle meets the parabola, and draw $Q'X'R'$ parallel to the axis, meeting the circle in X' and the tangent at P in R' ; then

XX' is parallel to RPR' (*Prob. 17*)

and when the circle becomes the circle of curvature,

WX' is parallel to RPR' ,

$\therefore WX'$ is at right angles to PU .

Now if the points Q and U coincide,

$X'U$ and PW will be equal,

and WX' will bisect PU ,

and $PWUX'$ will be a square,

$$\begin{aligned} \therefore \text{the } \angle GPS &= \text{the } \angle GPW, \\ &= \text{the } \angle GPN, \end{aligned}$$

$\therefore P$ is at the extremity of the latus rectum.



PROBLEMS ON THE ELLIPSE.

1. DRAW the diameter CD parallel to the tangent at P , meeting $S'P$ in E . Also draw the normal PGF perpendicular to CD , and meeting it in F .

Then the angle $SPS' = 2$ the angle $S'PG$.

Now since PE is always equal to AC , it is evident that the angle $S'PG$ will be greatest when PF is least.

But $PF \cdot CD = AC \cdot BC$ (*Prop. XXII. Cor.*)

$\therefore PF$ is least when CD is greatest, that is, when $CD = AC$ and P is at the extremity B of the minor axis.

2. Draw the latus rectum LSL' ; then

$$AC^2 : BC^2 :: AS \cdot A'S : SL^2 \text{ (Prop. XIII.)}$$

$$\text{or } AC^2 : BC^2 :: \cancel{AC^2} - CS^2 : SL^2 \text{ (Prop. IV.)}$$

$$\text{or } AC^2 : BC^2 :: BC^2 : SL^2,$$

$$\text{or } AC : BC :: BC : SL,$$

$$\therefore AA' : BB' :: BB' : LL',$$

$$\therefore LL' \text{ is a 3d proportional to } AA' \text{ and } BB'.$$

3. Draw BH parallel to LS , meeting the major axis in H ; then

$$\begin{aligned} \triangle BCH : \triangle LSS' &:: BC^2 : SL^2 \\ &:: AC^2 : BC^2. \text{ (Prob. 2.)} \end{aligned}$$

Take $CK = \frac{1}{2}CH$; then

$$\text{rectangle } BB' \cdot CK = \triangle BCH,$$

$$\therefore BB' \cdot CK : \triangle LSS' :: AA'^2 : BB'^2.$$

4. Let the tangent at L to one of the ellipses meet the minor axis produced in t ; then

$$Ct \cdot SL = BC^2 \text{ (Prop. XIV.)}$$

$$\therefore Ct : BC :: BC : SL$$

$$:: AC : BC \text{ (Prob. 2.)}$$

$$\therefore Ct = AC.$$

Hence Ct is constant, and the tangents will all pass through t .

5. In this problem the ellipses are to have their major axes equal as well as a common focus.

Let S be the given focus, and YPY' the given fixed line (see fig. *Prop. XV.*); then since S and YPY' are both given in position, the point Y of the perpendicular SY will be a fixed point. •

Now $YC = AC$ (*Prop. XV.*)

Hence the locus of C is a circle whose centre is Y and radius AC .

6. Draw SY , $S'Y'$ perpendicular to the given line (see fig. *Prop. XV.*), and take BC a mean proportional between SY and $S'Y'$; then

BC is the semi-minor axis. (*Prop. XV.*)

Also, since $AC^2 = BC^2 + CS^2$,

AC is also known; and since

$$CX : CA :: CA : CS \text{ (Prop. II.)}$$

the position of the directrix is known, and the ellipse may be described.

7. See fig. *Prop. IX.*

$$TQ : PN :: AT : AN,$$

$$T'Q' : PN :: A'T : A'N,$$

Again,

$$CT \cdot CN = CA^2,$$

$$\therefore CT : CA :: CA : CN,$$

$$\begin{aligned}
 CT + CA : CT - CA &:: CA + CN : CA - CN, \\
 \text{or } A'T : AT &:: A'N : AN, \\
 \therefore A'T : A'N &:: AT : AN,
 \end{aligned}$$

Hence

$$\begin{aligned}
 TQ : PN &:: TQ' : PN, \\
 \therefore TQ &= TQ'.
 \end{aligned}$$

8. See fig. *Prop.* IV.If SBS' be a right angle, SC and CB are equal,

$$\therefore AC^2 = SB^2 = SC^2 + BC^2 = 2BC^2.$$

9. Draw the ordinates QM, PN ; then

$$\begin{aligned}
 QM^2 : AM^2 &:: PN^2 : CN^2, \\
 \therefore A'M \cdot A'M : AM^2 &:: AC^2 - CN^2 : CN^2, \\
 \text{or } A'M : AM &:: AC^2 - CN^2 : CN^2, \\
 \therefore 2AC : AM &:: AC^2 : CN^2, \\
 \therefore 2AO : AQ &:: AO^2 : CP^2, \\
 \text{or } 2AO^2 : AO \cdot AQ &:: AO^2 : CP^2, \\
 \therefore AO \cdot AQ &= 2CP^2.
 \end{aligned}$$

10. If AB and CD are equally inclined to the axis, the diameters parallel to them will also be so, and will be therefore equal. Hence if AB and CD or these lines produced meet in O ,

$$OA \cdot OB = OC \cdot OD.$$

Hence a circle may be described about A, B, C, D ; and if AC and BD or these lines produced intersect in O' ,

$$O'A \cdot O'C = O'B \cdot O'D,$$

$\therefore AC$ and BD are equally inclined to the axis.

So with regard to AD and BC .

11. Draw PM, pm at right angles to the directrix. Produce MP, QS to meet in Q , and mp, qS to meet in q' ; then

$$Q'P : PM :: SA : AX,$$

$$\therefore Q'R = SP,$$

$\therefore SQ$ bisects the angle PSA' ;

So Sq' bisects the angle PSA' ,

$\therefore Q'Sq'$ is a right angle.

\therefore also $Q'Sq$ is a right angle.

12. Draw the diameter SK cutting the tangent at B at right angles in O ; then

$$SK \cdot SO = SB^2,$$

$$\text{or } SK \cdot BC = AC^2.$$

13. Through Q draw RQR' touching the inner circle in Q , and cutting the outer in R and R' ; then

$$PQ \cdot P'Q = QR^2,$$

$$= CR^2 - CQ^2,$$

$$= AC^2 - BC^2 = CS^2.$$

14. Draw the ordinate PN , and the semi latus rectum SL , then by similar triangles

$$SL : SN :: SG : SP,$$

$$:: SA : AX;$$

Also $SP : XN :: SA : AX,$

$$\therefore PL : SX :: SA : AX;$$

But $SL' : SX :: SA : AX,$

$$\therefore PL = SL'.$$

15. Draw the normals PG, DG' ; then

$$PG^2 : CD^2 :: BC^2 : AC^2 \text{ (Prop. XXIV.)}$$

and $DG'^2 : CP^2 :: BC^2 : AC^2,$

$$\therefore PG^2 + DG'^2 : CP^2 + CD^2 :: BC^2 : AC^2.$$

But $CP^2 + CD^2 = AC^2 + BC^2.$

$$\therefore PG^2 + DG'^2 : AC^2 + BC^2 :: BC^2 : AC^2,$$

$$\therefore PG^2 + DG'^2 \text{ is constant.}$$

16. Produce PG to meet CD in F , and join CP ; then

$$\begin{aligned} CQ^2 &= CP^2 + PQ^2 - 2PF \cdot PQ \text{ (Euclid, II. 13)} \\ &= CP^2 + 2CD \cdot PF - 2PF \cdot CD, \\ &= AC^2 + BC^2 - 2AC \cdot BC \text{ (Props. XX. XXII.)} \\ \therefore CQ &= AC - BC. \end{aligned}$$

17. Let the tangents at P and P' intersect at right angles in O . Draw CK, CK' at right angles respectively to OP and OP' ; then

$$CO^2 = CK^2 + CK'^2.$$

Again, from S draw SY and SY' perpendicular respectively to OP and OP' ; and join CY, CY' . Also let SY' and CK intersect in H ; then

$$\begin{aligned} CK^2 + SH^2 &= CY^2 = AC^2, \\ \text{and } CK'^2 + CH^2 &= CY'^2 = AC'^2, \\ \therefore CK^2 + CK'^2 + CS^2 &= 2AC^2, \\ \therefore CK^2 + CK'^2 &= AC^2 + BC^2, \\ \therefore CO^2 &= AC^2 + BC^2. \end{aligned}$$

18. Draw the ordinate QM ; then

$$\begin{aligned} NR : AN &:: QM : AM, \\ \text{and } NS : A'N &:: QM : A'M, \\ \therefore NR \cdot NS : AN \cdot A'N &:: QM^2 : AM \cdot A'M. \\ \text{But } PN^2 : AN \cdot A'N &:: QM^2 : AM \cdot A'M, \\ \therefore PN^2 &= NR \cdot NS. \end{aligned}$$

19. By similar triangles

$$\begin{aligned} GL : PN &:: SG : SP, \\ &:: CS : CA \text{ (Prop. XI.)} \end{aligned}$$

20. Since the tangent at L meets the directrix in X ,

$$\begin{aligned} QN : XN &:: SL : SX, \\ &:: SA : AX; \end{aligned}$$

Also

$$\begin{aligned} SP : XN &:: SA : AX, \\ \therefore QN &= SP. \end{aligned}$$

21. See fig. *Prop.* VII.

PZ bisects the $\angle SPW$,
 \therefore the $\angle SPZ$ is $>$ the $\angle MPZ$.

22. See fig. *Prop.* XXIV.

Let $S'P$ meet CD in E . Produce PF to meet the axis minor in K , and join EK ; then

$$\begin{aligned} PK : PG &:: NC : NG, \\ &:: AC^2 : BC^2 \text{ (Prop. XII.)} \\ \therefore PK \cdot PF &: PF \cdot PG :: AC^2 : BC^2, \\ \therefore PK \cdot PF &= AC^2 \text{ (Prop. XXIV.)} \\ &= PE^2 \text{ (Prop. XV. Cor.)} \\ \therefore \text{the } \angle PEK &\text{ is a right angle.} \end{aligned}$$

So also if PS produced meet CD' in E' , and $E'K$ be joined, the angle $PE'K$ may be proved to be a right angle.

\therefore the perpendiculars EK and $E'K$ meet the minor axis in the same point.

$$\begin{aligned} 23. \quad NG : NC &:: BC^2 : AC^2 \text{ (Prop. XII.)} \\ \therefore CT \cdot NG &: CT \cdot CN :: BC^2 : AC^2, \\ \therefore CT \cdot NG &= BC^2 \text{ (Prop. IX.)} \\ \therefore CT : BC &:: BC : NG. \end{aligned}$$

24. A circle may be described about $PYSN$;

$$\begin{aligned} \therefore \angle PNY &= \angle PSY. \\ \text{So} \quad \angle PNY' &= \angle PS'Y', \\ \text{and } \angle PSY &= \angle PS'Y', \\ \therefore \angle PNY &= \angle PNY', \\ \therefore PY : PY' &:: NY : NY'. \end{aligned}$$

25. See fig. *Prop.* XV.

Draw CZ at right angles to the tangent meeting $S'P$ in K ; then by similar triangles

$$KE : CE :: SP : PY.$$

But

$$CE = PY,$$

$$\therefore KE = SP.$$

26. Let R be the point where the circle described about QPQ' meets the ellipse; join PR meeting QQ' in O , and draw the diameter CP' parallel to PR , and CD parallel to $Q'Q'$; then from the ellipse

$$QO \cdot OQ' : PO \cdot OK :: CD^2 : CP'^2 \text{ (Prop. XXIX.)}$$

and from the circle

$$QO \cdot OQ' = PO \cdot OK,$$

$$\therefore CP = CD,$$

\therefore , CP' being fixed in position, PR , which is drawn parallel to CP' , meets the ellipse in a fixed point.

27. Produce QP' , $Q'P'$ to meet the tangent at P in R and R' . Draw the diameter CD parallel to the tangent at P ; also draw the diameters CE , CE' parallel respectively to QP' and $Q'P'$; then it is evident from Prop. XXIX. that

$$RP \cdot RQ : RP^2 :: CE^2 : CD^2,$$

$$\text{and } R'P \cdot R'Q' : R'P^2 :: CE'^2 : CD^2.$$

But

$$CE^2 = CE'^2,$$

Since PQ and $P'Q'$ are equally inclined to the axis,

$$\therefore RP \cdot RQ : R'P \cdot R'Q' :: RP^2 : R'P^2$$

But

$$RP : R'P :: RP' : R'P',$$

since $P'P$ bisects the $\angle Q'P'R$,

$$\therefore RP' \cdot RQ : R'P' \cdot R'Q' :: RP^2 : R'P^2$$

$$\therefore RQ : R'Q' :: RP : R'P,$$

$$\therefore QQ' \text{ is parallel to } RR'.$$

28. See fig. Prop. XXII.

Let a parallelogram $LPIL'I'$ be formed by drawing tangent at the extremities of a pair of conjugate diameters PCP' , and DCD' .

On the portion of the ellipse DPD' , take any point Q , and at the extremities of the diameter QOQ' , draw the tangents $RQr, R'Q'r'$ meeting the tangents at D and D' in R, R' and r, r' respectively. Also let Rr intersect Ll in O .

Now, since Ll is bisected in P , it is evident, that on whichever side of P the point O is taken, the parallelogram RR' is greater than the parallelogram LL' , and that the excess is twice the difference of the unequal triangles ROL, rOL .

Hence the parallelogram formed by drawing tangents at the extremities of conjugate diameters, (which is, of constant area) is the least which can be described about an ellipse.

$$29. (CP + CD)^2 = CP^2 + CD^2 + 2CP \cdot CD, \\ = AC^2 + BC^2 + 2CP \cdot CD \text{ (Prop. XX.)}$$

Hence $(CP + CD)^2$ is least when $CP \cdot CD$ is least.

Now $CP \cdot CD$ is $> PF \cdot CD$, unless $PF = CP$,
and $PF \cdot CD = AC \cdot BC$ (Prop. XXII. Cor.)

Hence $CP \cdot CD$ is $> AC \cdot BC$, unless $PF = CP$;
and when $PF = CP$,
 $CP \cdot CD = AC \cdot BC$;

$\therefore CP \cdot CD$ is least, when PF and CP are equal, or when CP and CD coincide with the axes of the ellipse.

30. Let S be the given focus, and P the given point. Join SP , and on SP produced make SW equal to the major axis AA' .

With centre P and radius PW , describe a circle WU . The other focus of the ellipse must be on this circle.

Now, since in an ellipse

$$CS^2 = AC^2 - BC^2 \text{ (Prop. IV.)} \\ \therefore SS'^2 = AA'^2 - BB'^2.$$

Hence we have the following construction for finding S' the other focus and centre.

On SW as diameter describe a semicircle, and in it place WH equal to the minor axis BB' .

With centre S and radius SH describe a circle cutting the circle WU in S' .

$$\begin{aligned}\text{Then} \quad SS'^2 &= SH^2 = SW^2 - WH^2 \\ &= AA'^2 - BB'^2.\end{aligned}$$

$\therefore S'$ is the other focus.

Bisect SS' in C ; then C is the centre.

31. Let Q be any point on the ellipse, and let QP produced meet CD in M , and let QP' meet CD in N . Also draw QV parallel to CD , and QR parallel to CP .

$$\begin{aligned}\text{Then} \quad CM : CP &:: QV : PV, \\ CN : CP' &:: QV : P'V, \\ \therefore CM \cdot CN : CP^2 &:: QV^2 : PV \cdot P'V, \\ &:: CD^2 : CP^2 \text{ (Prop. XXI.)} \\ \therefore CM \cdot CN &= CD^2.\end{aligned}$$

32. Let CP and CD be conjugate diameters.

$$\begin{aligned}\text{Then} \quad SP : XN &:: SA : AX \text{ (See fig. Prop. IX.)} \\ &:: CA : CX \text{ (Prop. II.)}\end{aligned}$$

$$\begin{aligned}\therefore SP : AC &:: XN : CX, \\ \therefore AC - SP : AC &:: CN : CX, \\ \text{or } AC - SP : CN &:: AC : CX.\end{aligned}$$

So also if DN' be the ordinate of D ,

$$\begin{aligned}AC - SD : CN' &:: AC : CX, \\ \therefore (AC - SP)^2 + (AC - SD)^2 &:: CN^2 + CN'^2 :: CA^2 : CX^2, \\ &:: CS^2 : CA^2;\end{aligned}$$

$$\text{But } CN^2 + CN'^2 = CA^2 \text{ (Prop. XX.)}$$

$$\therefore (AC - SP)^2 + (AC - SD)^2 = CS^2.$$

33. Draw the ordinates PN , DR , and produce them to meet the auxiliary circle in P' , D' ; also let CB produced meet the circle in B' .

Join $B'D'$, $B'P'$; $A'P'$, AD' , and let $A'P'$, AD' intersect in O' .

Now, since $P'D'$ is a quadrant of a circle,
the arc $AP' =$ the arc $B'D'$,
 $\therefore B'P'$ is parallel to AD' ;
So also $B'D'$ is parallel to $A'P'$,
 $\therefore B'O'$ is a parallelogram.

And since BP and $B'P'$ produced meet the axis in the same point, and $B'P'$ is parallel to AD' ,
 and $BC : B'C :: DR : D'R$,
 $\therefore BP$ is parallel to AD ;
 So also BD is parallel to $A'P$,
 $\therefore OB$ is a parallelogram.

Again, join $PD, P'D'$. These lines will, when produced, meet the axis in the same point. Now if any straight line be drawn at right angles to AA' , meeting $PD, P'D'$, it is easily seen that the portions of this line intercepted by the triangles $PBD, P'B'D'$ respectively are in the proportion of BC to AC .

Hence the triangles PBD , $P'B'D'$ are also in ~~this~~ proportion.

$$\therefore ODBP : O'D'B'P' :: BC : AC.$$

Now since $P'D'$ is a quadrant, the triangle $P'B'D'$ will have its greatest value when B' is half way between P' and D' , for the perpendicular from B' on $P'D'$ will then be greatest.

Hence $\angle ODBP$ is greatest when CP and CD are equal.

34. $PS \cdot Sp : AS \cdot A'S :: CQ^2 : AC^2$ (*Prop. XXIX.*)
 $\therefore PS \cdot Sp : CQ \cdot Cq :: AS \cdot A'S : AC^2,$
 $:: CA^2 - CS^2 : AC^2,$
 $:: BC^2 : AC^2.$

35. See fig. *Prop.* XIX.

$$\begin{aligned}
 AT : AC &:: PN : CN, \\
 &:: CR : DR \text{ (Prop. XIX. Cor.)} \\
 &:: AC : At, \\
 \therefore AT \cdot At &= AC^2.
 \end{aligned}$$

36. Draw the diameter Cp , Cq , Cr parallel to the tangents; then

$$\begin{aligned}
 P'R^2 : P'Q^2 &:: Cr^2 : Cq^2 \text{ (Prop. XXIX.)} \\
 \text{or } P'R &: P'Q :: Cr : Cq. \\
 \text{So } Q'P : QR &:: Cp : Cr, \\
 \text{and } R'Q : RP &:: Cq : Cp, \\
 \therefore P'R \cdot QP \cdot R'Q &= P'Q \cdot QR \cdot RP.
 \end{aligned}$$

37. See fig. Prop. IX.

Let the tangent at P meet the tangent at A in Q ; then QP and QA subtend equal angles at S (Prop. XVII.)

$\therefore QS$ bisects the $\angle TSP$,
and PT bisects the $\angle SPW$,
 $\therefore Q$ is the centre of the circle ST , SP and $S'P$ produced,
 \therefore the locus required is the tangent at the vertex A .

38. It is easily seen that this question is only the converse of Prob. 33.

39. Let ACA' , aCa' be the major axes of the ellipses, inclined to one another at any angle.

Let P be the point where the ellipses intersect within the angle ACA , and Q the point within the angle $A'Ca$; then

since CP is a common semi-diameter of both ellipses,

$\therefore CP$ makes equal angles with CA and Ca ,

$\therefore CP$ bisects the angle ACA ,

So CQ bisects the angle $A'Ca$,

\therefore the $\angle PCQ$ is half of ACA and $A'Ca$,

$\therefore PCQ$ is a right angle.

40. Draw the ordinates PM , QN ; then

$$\begin{aligned} PM : QN &:: SP : SQ, \\ &:: XM : XN, \\ \therefore PM : XM &:: QN : XN, \\ \therefore \angle PXM &= \angle QXN. \end{aligned}$$

41. Draw the ordinates PM , QN ; and let Cp , Cq be the diameters parallel to the tangents at P and Q respectively; then

$$RP \cdot R'P : RQ \cdot R'Q :: XM \cdot X'M : XN \cdot X'N.$$

$$\text{Now } XM : SP :: CA : CS :: XN : SQ,$$

$$\text{and } X'M : S'P :: CA : CS :: X'N : S'Q,$$

$$\therefore XM \cdot X'M : SP \cdot S'P :: XN \cdot X'N : SQ \cdot S'Q,$$

$$\begin{aligned} \therefore RP \cdot R'P : RQ \cdot R'Q &:: SP \cdot S'P : SQ \cdot S'Q, \\ &:: Cp^2 : Cq^2 \text{ (Prop. XXIV.)} \\ &:: OP^2 : OQ^2 \text{ (Prop. XXIX.)} \end{aligned}$$

42. See fig. Prop. XIV.

Let the normal to the ellipse at P meet CQ produced in R' ; then

the triangles $R'PR$ and TPQ are similar.

$$\begin{aligned} \therefore R'R : TQ &:: PR : QP, \\ &:: CQ : TQ, \\ \therefore R'R &= CQ = AC, \\ \therefore CR' &= CR + AC, \\ &= BC + AC \text{ (Prop. XIV.)} \end{aligned}$$

\therefore the locus of R' is a circle, whose centre is C and radius $AC + BC$.

43. See fig. Prop. XV.

Let CD meet SP produced in E , and $S'P$ in E' ; then

$$\begin{aligned} S'E' &= S'P - PE' = S'P - AC, \\ \text{and } SE &= PE - SP = AC - SP. \end{aligned}$$

But

$$\begin{aligned} S'P + SP &= 2AC, \\ \therefore S'P - AC &= AC - SP, \\ \therefore SB &= S'E'. \end{aligned}$$

Again, if D and D' be the diameters of the circles described about SCE and $S'CE'$, since the perpendiculars from S and S' upon CQ are equal,

$$\begin{aligned} \therefore D : D' &:: CS : SE :: CS' : S'E' \text{ (Euclid VI. C.)} \\ \therefore D &= D'. \end{aligned}$$

44. Let PSP' be any focal chord, and O its middle point. Join CO and produce it to meet the ellipse in Q . Also draw the tangent QT , and the ordinate QN .

Since

 O is the middle point of PP' , $\therefore QT$ is parallel to PP' ,

$$\therefore OM : MC :: QN : NC,$$

$$\text{and } OM : MS :: QN : NT,$$

$$\therefore OM^2 : MC \cdot MS :: QN^2 : NC \cdot NT,$$

$$:: BC^2 : AC^2 \text{ (Prop. XIII.)}$$

\therefore the locus of O is an ellipse, similar to the given ellipse, described upon CS as major axis.

45. Let the circle cut the minor axis in K ; then since the angle BSK is a right angle,

$$\begin{aligned} \therefore BK \cdot BC &= SB^2, \\ &= AC^2. \end{aligned}$$

But if BU be the chord of the circle of curvature at B through C , and therefore in this case the diameter,

$$BU \cdot BC = 2AC^2 \text{ (Prop. XXVI.)}$$

$$\therefore 2BK = BU,$$

$$\therefore BK = BO,$$

$\therefore K$ coincides with the centre of the circle of curvature.

46. See fig. Prop. XI.

PROBLEMS.

Draw SO bisecting the angle $\dot{PSS'}$, and meeting PG in O ; then O is the centre of the circle.

Draw the ordinate OM at right angles to AC .

$$\begin{aligned}\text{Now} \quad MG : NG &:: OG : PG, \\ &:: SG : SG + SP, \\ &:: CS : CS + AC \text{ (Prop. XI.)}\end{aligned}$$

$$\text{Also} \quad NG : NC :: CA^2 - CS^2 : AC^2 \text{ (Prop. XII.)}$$

$$\therefore MG : NC :: (CA - CS) CS : AC^2,$$

$$:: CA \cdot CS - CS^2 : AC^2,$$

$$\text{and } CG : NC :: CS^2 : AC^2,$$

$$\therefore MC : NC :: CA \times CS : AC^2,$$

$$:: CS : AC,$$

$$\therefore MC : CS :: NC : AC,$$

$$\therefore SM : CS :: AN : AC,$$

$$\text{and } S'M : CS :: A'N : AC,$$

$$\therefore SM \times S'M : CS^2 :: AN \times A'N : AC^2,$$

$$:: PN^2 : BC^2 \text{ (Prop. XIII.)}$$

$$\therefore SM \cdot S'M : PN^2 :: CS^2 : CA^2 - CS^2.$$

$$\text{Also} \quad PN^2 : OM^2 :: (CA + CS)^2 : CS^2,$$

$$\therefore SM \cdot S'M : OM^2 :: (CA + CS)^2 : (CA - CS)^2,$$

$$:: A'S^2 : AS^2.$$

Hence the locus of O is an ellipse, whose major is SS' .

Also if Cb be the semi-minor axis,

$$Cb : CS :: AS : A'S.$$

47. Let P, P', Q, Q' be the four points of intersection.

Join $PQ, P'Q'$, and let these lines or these produced intersect in O .

Draw the semi-diameters Cp, Cp' parallel respectively to PQ and $P'Q'$.

$$\text{Then } PO \cdot OQ : P'O \cdot OQ' :: Cp^2 : Cp'^2 \text{ (Prop. XXIX.)}$$

But $PO \cdot OQ = P'O \cdot O'Q$ from the circle,

$$\therefore Cp = Cp'.$$

48. Let OP, OP' be the two lines at right angles to each other, and C the centre of the ellipse in any position; then, as in *Prob. 17*,

$$OC^2 = AC^2 + BC^2,$$

\therefore the locus of C is the arc of a circle whose centre is O .

49. See fig. *Prop. XXIV*.

Draw SM at right angles to CD , and let MS and CP be produced to meet in Q . Also draw QH at right angles to the axis.

Now

$$CH : CN :: CQ : CP,$$

$$:: CM : CF,$$

$$\therefore CS : CG,$$

$$\therefore CH : CS :: CN : CG,$$

$$:: CX : CS \text{ (Prop. XII.)}$$

$$\therefore CH = CX,$$

\therefore the point Q is on the directrix.

50. For *radius* read *diameter*.

If O be the centre of the circle of curvature at I ,

$$PO \cdot PF = CD^2 \text{ (Prop. XXVII.)}$$

$$\therefore PO : CD :: CD : PF,$$

$$:: CD^2 : PF \cdot CD,$$

$$:: CD^2 : AC \cdot BC \text{ (Prop. XXII. Cor.)}$$

Hence if

$$CD^2 = AC \cdot BC,$$

$$\therefore PO = CD,$$

$$\therefore \text{also } PO^2 = AC \cdot BC,$$

$\therefore PO$ is a mean proportional between AC and BC at the same time that CD is.

Hence, if a circle be described with centre C and radius

a mean proportional between AC and BC , intersecting the ellipse in D , and CP be drawn parallel to the tangent at D , P will be the point required.

51. See *Prop.* XXVI.

If $CP = CD$, then

$$PH \cdot CP = 2 \cdot CP^2, \\ \therefore PH = 2 \cdot CP = PP',$$

\therefore the circle of curvature meets the ellipse at the point P .

52. Let the lines drawn at right angles to AP , $A'P$ meet in Q , and draw QM at right angles to AA' ; then

$$QM : AM :: AN : PN, \\ \text{and } QM : A'M :: A'N : PN, \\ \therefore QM^2 : AM \cdot A'M :: AN \cdot A'N : PN^2, \\ \therefore AC^2 : B'C^2 \text{ (Prop. XIII.)}$$

\therefore the locus of Q is an ellipse, similar to and concentric with the given ellipse, and having AA' for its minor axis.

53. Let C and c be the centres of the two ellipses, which intersect one another in the points P, Q, R, S . Join PQ, RS ; and let these lines, or these lines produced, intersect one another in the point O .

From C and c draw the semi-diameters CU, CV parallel to PQ , and cu, cv parallel to RS ; then

$$PO \cdot OQ : RO \cdot OS :: CU^2 : CV^2 \text{ (Prop. XXIX.)} \\ \text{and } RO \cdot OQ : RO \cdot OS :: cu^2 : cv^2 \text{ (ditto.)} \\ \therefore CU : CV :: cu : cv.$$

Now if the ellipses have their major axes at right angles to one another, it is evident that since the angles UCV and ucv are equal and CU, cu parallel,

$$CU > = < CV, \\ \text{according as } cv > = < cu \\ \text{and } \therefore \text{ the only case in which the proportion}$$

$$CU : CV :: cu : cv$$

can hold good is when CU and CV are equal.

Again, if the major axes of the ellipses are parallel, the only cases in which the proportion can hold good is when the ellipses are similar, or as before when CU and CV are equal.

But if the ellipses are similar, and their major axes parallel, they can only intersect in two points.

Hence, whether the major axes are at right angles or parallel, we must have

$$CU = CV.$$

$$\therefore PO \cdot OQ = RO \cdot OS.$$

And a circle may therefore be described through P, Q, R, S .

54. Let VCV' be the fixed diameter of the circle, along which CN is measured; and draw the diameter DCD' parallel to NQ ; then, since NQ makes a constant angle with NP , the diameter DCD' is also fixed.

Now $PN^2 = VN \cdot V'N$ from the circle,

$$\therefore QN^2 = VN \cdot V'N$$

$$\text{or } QN^2 : VN \cdot V'N :: CD^2 : CV^2.$$

\therefore the locus of Q is an ellipse, of which CV and CD are the *equal* conjugate semi-diameters.

The position and length of the *equal* conjugate diameters being now known, the ellipse may be constructed as follows,

Draw ACA' bisecting the acute angle formed by VCV' , DCD' , and BCB' bisecting the supplementary angle formed by the same lines.

Then ACA' , BCB' represent the *directions* of the major and minor axes of the ellipse respectively.

Draw VM at right angles to CA , and Cv bisecting the angle ACB , and let these lines meet in v ; then

the point v is on the *auxiliary* circle.

Take CA equal to Cv , and
 make $BC : AC :: VM : CM$;
 then AC and BC are the semi-major and minor axes.

55. It is evident from the symmetry that the two ellipses will have a common centre, and that that centre must coincide with the point where the diameters of the parallelogram intersect.

Also since the ellipses are supposed to have their major axes of the same length, their auxiliary circles will also coincide.

Now since any side of the parallelogram is a common tangent to both ellipses, and by *Prop. XV.* the feet of the perpendiculars from the foci upon the tangent lie on the circumference of the auxiliary circle, it is evident, that the perpendiculars from the foci of one of the ellipses upon the sides of the parallelogram, will coincide with the perpendiculars from the foci of the other ellipse upon the same lines.

Or, in other words, the lines joining the foci of the ellipses will be at right angles to the sides of the given parallelogram. These lines will therefore form a parallelogram equiangular to the given parallelogram.

56. Let H be the common focus of the ellipses, and S, S' their other foci. •

Then, since the major axes of the ellipses are equal, any point where they intersect must be equidistant from S and S' .

Any point, therefore, where the ellipses intersect must lie upon the line bisecting SS' at right angles, for this line contains *all* the points, whose distances from S and S' are equal to each other.

And since this line can only intersect either ellipse in *two* points, it is evident that the ellipses themselves can only intersect in two points.

57. Since arc $SR = \text{arc } S'R$
 $\therefore PR$ bisects the angle SPS' ,
 and coincides with the normal PQ produced.

Hence the triangles $S\dot{P}R$, QPS' are similar,

$$\begin{aligned}\therefore SR : PR &:: S'Q : S'P, \\ &:: CS : CA \text{ (Prop. XI.)} \\ &:: CS : CA : AC^2;\end{aligned}$$

But $PR : PQ :: PC : NC,$
 $\therefore AC^2 : BC^2 \text{ (Prop. XII.)}$
 $\therefore SR : PQ :: CS : CA : BC^2.$

58. Draw SO bisecting the angle PSS' , and meeting the normal PQ in O ; then O is the centre of the inscribed circle. Draw OM perpendicular to the axis; then

$$\begin{aligned}OM : PN &:: OQ : PQ, \\ &:: SQ : SQ + SP, \\ &:: CS : CS + CA,\end{aligned}$$

$$\therefore OM : CS :: PM : CS + CA,$$

Also $2R : S'P :: SP : PN$ (Euclid VI. B.)

$$\therefore 2R : r : CS : S'P :: SP : CS + CA,$$

or $2R : r : SP : S'P :: CS : CS + CA.$

59. Join QK , GL and let KL meet PQ in O ; then, since the angles at K and L are right angles, and the angle SPS' is bisected by PQ ,

\therefore the triangles PGL , PGK are equal in all respects.

Again, since OP , PL are equal to OP , PK ,

and the $\angle OPL =$ the $\angle OPK$,

$$\therefore OL = OK,$$

and the $\angle POL =$ the $\angle POK$,

$\therefore PQ$ is at right angles to KL .

60. See fig. Prop. XV.

Draw $S'O$ parallel to SP meeting YS produced in O ; then

$$\begin{aligned}SO : SP &:: SW : WI, \\ &:: SP + S'P : SP,\end{aligned}$$

$$\therefore S'O = SP + S'P = 2AC$$

\therefore the locus of O is a circle, whose centre is S' and radius $2AC$.

61. Draw the conjugate diameter CD parallel to TPt . Also draw the ordinate DR ; then

$$\begin{aligned} TP : CD &:: PN : DR, \\ &:: CR : CN \text{ (Prop. XIX. Cor.)} \\ &:: CD : Pt, \end{aligned}$$

$$\therefore TP \cdot Pt = CD^2.$$

But $CP \cdot PL = TP \cdot Pt,$

$$\begin{aligned} \therefore 2CP \cdot PL &= 2CD^2, \\ &= PH \cdot CP \text{ (Prop. XXVI.)} \\ \therefore 2PL &= PH. \end{aligned}$$

Again, $CP \cdot CL = CP \cdot PL + CP^2,$

$$\begin{aligned} &= CD^2 + CP^2, \\ &= AC^2 + BC^2 \text{ (Prop. XX.)} \end{aligned}$$

62. Since CR' bisects PQ , the tangent at R' will be parallel to PQ (Prop. XVIII.)

So the tangents at P' and Q' will be parallel respectively to QR and PR .

\therefore the triangle rqp formed by the tangents at P', Q', R' is similar to the triangle PQR .

Now, since $P'Q'$ is equal and parallel to PQ ,

\therefore the triangle $rP'Q' =$ the triangle PQR .

So the triangle $qP'R' =$ the triangle PQR ,

and the triangle $pQ'R' =$ the triangle PQR .

\therefore the triangle $pqr = 4 \cdot$ the triangle PQR .

63. See fig. Prop. XV.

Draw the ordinate PN ; then PN is a common chord of both circles.

Now $\angle NS'J = \angle NY'J = \angle Y'NP = \angle Y'S'P$,

and $\angle ASI = \angle IYN = \angle YNP = \angle YSP$;

But $\angle YSP = \angle Y'S'P$,

$\therefore \angle NS'J = \angle ASI$,

and $\angle CS' = \angle CS$;

$\therefore IS$ and JS' will, when produced, meet CB in the same point.

64. Draw the diameter RCR' , and from any point Q on the ellipse draw QR and QR' .

Also draw the diameters PCP' , DCD' parallel respectively to QR and QR' .

Now since C is the middle point of RR' ,

$\therefore PCP'$ bisects QR' ,

\therefore the tangent at P is parallel to QR' ,

and therefore to DCD' ;

$\therefore CD$ is conjugate to CP .

65. See fig. *Prop. XVI*.

If SP and SP' are in the same straight line, then the point O is on the directrix (*Prop. VIII*.)

and the $\angle OSP$ is a right angle (*Prop. VI*.)

\therefore the $\angle OMP$ is a right angle (*Prop. XVII*.)

so also the $\angle OM'P$ is a right angle,

\therefore the angles MOM' and $MS'M'$ are equal to two right angles.

But

$$MOM' = 2POP',$$

$\therefore 2T + O = 2$ right angles.

66. See fig. *Prop. XV*.

$$\begin{aligned} \triangle SPS' &= \triangle WSS' - WSP, \\ &= SY \cdot YY' - SY \cdot YP, \\ &= SY \cdot PY'. \end{aligned}$$

$$\begin{aligned} \text{Now} \quad & PY' : S'Y' :: PY : SY, \\ \therefore SY \cdot PY' : SY \cdot S'Y' :: PY : SY, \\ \text{or } \triangle SPS' : BC^2 :: PY : SY. \end{aligned}$$

So also if SY_1 be perpendicular to the tangent at Q ,
 $\triangle SQS' : BC^2 :: QY_1 : SY_1$.

But since the angles SPY , SQY , are complementary,
 $\therefore PY : SY :: SY_1 : QY_1$;
 $\therefore \triangle SPS' : BC^2 :: BC^2 : \triangle SQS'$.

The angles SPS' , SQS' are evidently supplementary.

The least value of either of the angles is zero, and the greatest $\angle SBS'$.

Hence the problem is evidently impossible,
 unless $2SBS'$ is > 2 right angles,
 or SBC is $> \frac{1}{2}$ a right angle,
 in which case $CS > BC$,
 or $BC < CS$.

67. Let CP be the fixed diameter; and CD the diameter which is conjugate to it in any one of the ellipses.

On CP produced take a fixed point O , and from it draw the tangent OQ to the ellipse to which CD belongs.

Draw QV parallel to CD , meeting CP in V , and produce PC to meet the ellipse in P' .

$$\begin{aligned} \text{Now} \quad & CV \cdot CO = CP^2 \text{ (Prop. XVIII.)} \\ \therefore & CV \text{ is constant,} \\ \therefore & PV \text{ and } P'V \text{ are also constant.} \end{aligned}$$

$$\text{Again, } QV^2 : PV \cdot P'V :: CD^2 : CP^2 \text{ (Prop. XXI.)}$$

$$\begin{aligned} \text{But} \quad & CD^2 = CP^2, \\ \therefore & QV^2 = PV \cdot P'V, \\ \therefore & QV \text{ is constant.} \end{aligned}$$

And since the point V is also fixed, therefore the locus of Q is a circle whose centre is V , and radius a mean proportional between PV and $P'V$.

68. See fig. *Prop. IV.*

Let AA' be one of the longer sides of the rectangle, and B' the intersection of the diagonals.

Produce AB' , $A'B'$ to meet the tangents to the ellipse, at A' and A in R' and R

Then RR' will be the side of the rectangle opposite to AA' .

Take any point P on the upper portion of the ellipse, and join PR , PR' meeting AA' in M and M' .

Also produce the ordinate NP to meet the auxiliary circle in Q .

By similar triangles

$$MM' : RR' :: PM : PR,$$

$$:: MN : AN,$$

$$\therefore MM' : MN :: AA' : AN.$$

Also $MN : AM :: PN : AR,$

$$:: PN : BB',$$

$$\therefore MM' : AM :: PN . AA' : AN . BB',$$

$$:: PN . AC : AN . BC.$$

But $PN : QN :: BC : AC$ (*Prop. XIII. (cor.)*)

$$\therefore PN . AC = QN . BC,$$

$$\therefore MM' : AM :: QN . BC : AN . BC,$$

$$:: QN : AN.$$

So also $MM' : A'M' :: QN : A'N,$

and $AN : QN :: QN : A'N,$

$$\therefore AM : MM' :: MM' : A'M'.$$

69. See fig. *Prop. XXIV.*

Draw SH perpendicular to PG ; then

$$PQ : PF :: SP : SH.$$

And if PM be the ordinate of P ,

$$PM : PG :: SH : SG,$$

$$\therefore PQ \cdot PM : PF \cdot PG :: SP : SG.$$

$$:: CA : CS. (Prop. XI.)$$

But $PF \cdot PG = BC^2$ (*Prop. XXIII.*)

$$\therefore PQ \cdot PM : BC^2 :: CA : CS,$$

$$\therefore PQ \text{ varies inversely as } PM.$$

70. See fig. *Prop. XIX.*

Draw QM at right angles to AA' ; then since the triangles QMS , CNP are similar,

$$\therefore QM : SM :: CN : NP.$$

So also $QM : S'M :: CR : DR$,

$$\therefore QM^2 : SM \cdot S'M :: CN \cdot CR : NP \cdot DR.$$

But $CN : DR :: AC : BC$ (*Prop. XIX. Cor.*)

$$\text{and } CR : NP :: AC : BC,$$

$$\therefore CN \cdot CR : DR \cdot NP :: AC^2 : BC^2,$$

$$\therefore QM^2 : SM \cdot S'M :: AC^2 : BC^2.$$

Hence the locus of Q is an ellipse, *similar* to and concentric with the given ellipse, and which has SS' for its minor axis.

PROBLEMS ON THE HYPERBOLA.

1. LET S and S' be the centres of the given circles, and P the centre of a circle described touching them both.

Join SP , $S'P$, and let these lines, or these lines produced, meet the circles of which S and S' are the centres in the points of contact Q and Q' ; then

✓ (1) If the given circles are touched *both* internally, or *both* externally,

$$\text{since } PQ = PQ'$$

$$\therefore SP \mp S'P = SQ \mp SQ',$$

and the locus of P is one branch of an hyperbola, of which S and S' are the foci; but

(2) If the given circles are touched, *one* externally, and the *other* internally,

$$SP + S'P = SQ + S'Q'$$

and the locus is an ellipse.

2. See fig. *Prop. VI.*

Let the tangent at P meet the tangents at A and A' , in U and U' ; join SU , $S'U'$; then

since UA and UP are tangents,

$$\therefore \text{the } \angle USA = \text{the } \angle USP. \text{ (Prop. XIV.)}$$

So the $\angle U'SA' = \text{the supplement of } \angle U'SP \text{ (Prop. XIV.)}$
 $= \text{the } \angle U'SQ,$

$$\therefore \text{the } \angle USA \text{ is half the } \angle ASP,$$

$$\text{and the } \angle U'SA \text{ is half the } \angle ASQ,$$

$$\therefore \angle USU' \text{ is a right angle.}$$

$\therefore S$ lies on the circle described upon UU' as diameter. So for the point S' .

8. If P be one of the points where the ellipse and hyperbola intersect, and S and S' the common foci; then

The line PT which bisects the $\angle SPS'$, is a normal to the ellipse, and a tangent to the hyperbola at the point P .

\therefore the curves intersect at right angles.

$$\begin{aligned}
 9. \quad & \underline{AR : PN :: AT : TN}, \\
 & \text{and } \underline{A'r : PN :: A'T : TN}, \\
 \therefore & AR.A'r : PN^2 :: AT.A'T : TN^2, \\
 & :: CA^2 - CT^2 : TN^2, \\
 & :: CT.CN - CT^2 : TN^2, \\
 & :: CT.TN : TN^2, \\
 & :: CT : TN, \\
 & :: \underline{CT.CN : CN.TN}, \\
 & :: \underline{AC^2 : AN.A'N} \text{ (Prop. X.)} \\
 \text{or } & AR.A'r : AC^2 :: PN^2 : AN.A'N, \\
 & :: BC^2 : AC^2 \text{ (Prop. X.)} \\
 \therefore & AR.A'r = BC^2 = AR'.A'r'.
 \end{aligned}$$

10. See fig. *Prop.* XXI.

Let LPl , $L'P'l'$ be the two tangents; then

$$CL.Cl = CS^2 = CL'.Cl' \text{ (Prop. XXII.)}$$

$$\therefore CL : CL' :: Cl' : Cl,$$

$$\therefore Ll' \text{ is parallel to } L'l.$$

11. See fig. *Prop.* XVII.

Since the $\angle CES$ is a right angle,

$$\therefore SE : CE :: AO : AC,$$

$$:: BC : AC,$$

$$\text{But } CE = AC \text{ (Prop. XVII.)}$$

$$\therefore SE = BC.$$

12. See fig. *Prop.* XVII.

$$\begin{aligned} CE : CO &:: CX : CA, \\ &:: CA : CS \text{ (Prop. II.)} \\ \therefore AE &\text{ is parallel to } SO. \end{aligned}$$

13. Since $CP^2 \cup CD^2 = AC^2 \cup BC^2$ (*Prop.* XXIV.)
 \therefore if $AC^2 = BC^2$
 $CP^2 = CD^2$.

14. $NG : NC :: BC^2 : AC^2$ (*Prop.* IX.)
 if $BC^2 = AC^2$,
 $NG = NC$,
 $\therefore PG = CP$.

15. See fig. *Prop.* XXV.

Join QP, QP' ; bisect QP, QP' in U and U' , and join CU, CU' ; then

since QP' is parallel to CU , and QP to CU' ,

$\therefore CU$ bisects the chords parallel to CU' ,

and CU' bisects the chords parallel to CU ;

$\therefore CU$ and CU' are in the directions of conjugate diameters.

But in a *rectangular* hyperbola the conjugate diameters are equal (*Prob.* 13), and are therefore equally inclined to either asymptote.

Hence QP and QP' , which are parallel to conjugate diameters, are also equally inclined to either asymptote.

16. See fig. *Prop.* XXIII.

Since $CPLD$ is a parallelogram, and that the diameters of a parallelogram bisect each other,

$\therefore PD$ is bisected by CL .

17. See fig. *Prop.* XVI.

Join AP , and produce it both ways to meet the lines drawn through A' parallel to CR and Cr in Q and Q' . Also let Q, Q' meet CR and Cr in V and U' ; then

$PV = AV$, and $PV' = AV'$ (*Prop.* XIX.)

But since AA' is bisected in U ,

$$\begin{aligned}QU &= AU, \text{ and } Q'U' = AU', \\ \therefore PQ &= UQ', \text{ and } PQ' = UQ, \\ \therefore PQ &= PQ'.\end{aligned}$$

18. See fig. *Prop. XXXII.*

$$\begin{aligned}SP &= PI \text{ (Prop. XXXII.)} \\ &= CR - AC \text{ (Prop. XVII.)}\end{aligned}$$

So also $S'P' = CR - BC$.

$$\therefore S'P' - SP = AC - BC.$$

19. Draw CZ parallel to SP , and DZ perpendicular to CZ ; also draw SY and $S'Y'$ perpendicular to the tangent at P ; then

the triangles DEC , SPY are similar.

$$\begin{aligned}\therefore DE : CD &:: SY : SP, \\ &:: S'Y' : S'P \text{ (Prop. XII.)} \\ \therefore DE^2 : CD^2 &:: SY \cdot S'Y' : SP \cdot S'P, \\ &:: BC^2 : CD^2 \text{ (Prop. XXXII.)} \\ \therefore DE &= BC.\end{aligned}$$

20. Let CP , CD be the given conjugate diameters.

Join PD , and bisect PD in H ; and join CH .

Also draw CK parallel to PD ; then

CH and CK are the asymptotes.

Bisect the angle HCK by the line CA , and draw CB at right angles to CA ; then

CA and CB are in the directions of the axes.

Draw the ordinate PN , and the tangent PT parallel to CD , meeting CN in T .

Take CA a mean proportional between CN and CT ; then

CA is the transverse semi-axis.

Also produce NP and CH to meet in R , and take BC a fourth proportional to RN , PN , and AC , so that

$$RN : PN :: AC : BC.$$

Then BC is the conjugate semi-axis.

21. Draw the ordinate PN ; then, since the hyperbola is rectangular,

$$\begin{aligned} PN^2 &= CN^2 - CA^2 \text{ (Prop. X.)} \\ \therefore PN^2 + AC^2 &= CN^2, \\ \text{or } AQ^2 &= PQ^2. \\ \therefore AQ &= PQ. \end{aligned}$$

22. See fig. *Prop. XXV.*

Let CP and CD be a pair of conjugate diameters, which will be equal, since the hyperbola is rectangular.

Let QV , drawn parallel to CD , pass through the focus S , and let it meet the hyperbola again in q .

Through S draw the chord $SQ'V'q'$ parallel to CP , meeting both branches of the hyperbola, and intersecting CD' in V' .

Draw the ordinates PN , $D'M'$ at right angles to the transverse axis, and let the tangents at P and D' meet this axis in the points T and T' respectively; then

$$CV : CP :: CS : CT.$$

But $CS \cdot CX = CA^2 = CN \cdot CT$ (*Props. II. and VIII.*)

$$\therefore CS : CT :: CN : CX,$$

Hence $CV^2 : CP^2 :: CN^2 : CX^2 \dots (1).$

Again $CV' : CD' :: CS : CT'.$

But $CT' \cdot CM' = CA^2$ (*Prop. XV.*)
 $= CS \cdot CX$ (*Prop. II.*)

$$\therefore CS : CT' :: CM' : CX,$$

Hence $CV'^2 : CD'^2 :: CM'^2 : CX^2 \dots (2).$

∴ from (1) and (2),

$$\begin{aligned} CV^2 - CV'^2 &: CP^2 :: CN^2 - CM'^2 : CX^2, \\ &:: CN^2 - PN^2 : CX^2 \text{ (Prop. XVI.)} \\ &:: AC^2 : CX^2 \text{ (Prop. X.)} \\ &:: CS^2 : CA^2 \text{ (Prop. II.)} \end{aligned}$$

But

$$\begin{aligned} CS^2 &= 2CA^2, \\ \therefore CV^2 - CV'^2 &= 2CP^2, \\ \therefore CV^2 - CP^2 &= CP^2 + CV'^2. \end{aligned}$$

But

$$QV^2 = CV^2 - CP^2 \text{ (Prop. XXVI.)}$$

and

$$\begin{aligned} Q'V'^2 &= CV'^2 + CP^2 \text{ (Prop. XXVI. Cor.)} \\ \therefore QV &= Q'V, \\ \text{or } Qq &= Q'q'. \end{aligned}$$

23. Since the hyperbola is rectangular,

$$\begin{aligned} \therefore PN^2 &= CN \cdot NT \text{ (Prop. X.)} \\ \therefore PN : NT &:: CN : PN, \\ \therefore \text{the angle } PTN &= \text{the angle } CPN, \\ \therefore \text{the angle } TCY &= \text{the angle } PCN. \end{aligned}$$

Hence

$$\begin{aligned} CY : CT &:: CN : CP, \\ \therefore CY \cdot CP &= CN \cdot CT, \\ &= CA^2 \text{ (Prop. VIII.)} \\ \therefore CY : CA &:: CA : CP, \\ \therefore \text{the triangles } CYA, CAP &\text{ are similar.} \end{aligned}$$

24. See fig. Prop. XVII.

Let Q be the centre of the circle.

Draw QF at right angles to CO , and let the latus rectum SL be produced to meet the asymptote CR in H ; then

$$\begin{aligned} QF : QC &:: CA : CO, \\ &:: BC : CS. \end{aligned}$$

$$\therefore QF : QF + QC :: B'C : BC + CS;$$

$$\text{or } QF : BC :: AC : B'S, \dots (1)$$

where S is the focus of the conjugate hyperbola (Art. 56).

$$\text{Again, } LS^2 : CS^2 - CA^2 :: BC^2 : AC^2, \text{ (Prop. X.)}$$

$$\text{or } LS^2 : BC^2 :: BC^2 : AC^2,$$

$$\therefore LS : BC :: BC : AC,$$

$$\text{and } HS : BC :: CS : AC,$$

$$\therefore HL : BC :: CS - BC : AC, \\ \therefore BS : AC \dots (2).$$

$$\text{Now } BS \cdot B'S = CS^2 - CB^2, \\ = AC^2,$$

$$\therefore AC : B'S :: BS : AC.$$

$$\text{Hence } QF : BC :: HL : BC,$$

$$\therefore QF = HL.$$

25. See fig. Prop. XIX.

Let the chords be drawn parallel to CR ; then

$$CH \cdot HR : RQ \cdot QR' :: CR^2 : RR'^2,$$

$$\text{or } CH \cdot CH' : PL^2 :: CR^2 : 4RV^2 \text{ (Prop. XX.)} \\ \therefore CL^2 : 4PL^2,$$

$$\therefore 4CH \cdot CH' = CL^2 = 4CK^2,$$

$$\therefore CH \cdot CH' = CK^2.$$

$$26. \quad RK : RP :: P'K' : P'R',$$

$$\text{but } PR = P'R' \text{ (Prop. XIX.)}$$

$$\therefore RK = P'K',$$

$$\text{so also } R'K' = PK.$$

27. Draw RM perpendicular to AA' produced; then

$$RM : PN :: AM : AN,$$

$$\text{and } RM : QN :: A'M : A'N,$$

$$\therefore RM^2 : PN \cdot QN :: AM \cdot A'M : AN \cdot A'N,$$

$$\text{but } PN \cdot QN = AN \cdot A'N,$$

$$\therefore RM^2 = AM \cdot A'M.$$

\therefore the locus of R is a rectangular hyperbola, of which AA' is the transverse axis.

28. See fig. *Prop. XXIII.*

Let P be one of the points in which the hyperbolas in the figure intersect the hyperbola whose axes are in the directions of CL and CL' ; then, since the hyperbolas are rectangular and equal, it is evident that

CP bisects the angle RCN .

$$\begin{aligned} \text{Now the angle } CPT &= \text{the angle } PCD, \\ &= 2 \cdot \text{the angle } PCH, \\ &= \text{the angle } RCN, \\ &= \text{half a right angle,} \end{aligned}$$

\therefore the angle between the tangents to the hyperbolas at P is a right angle.

29. Join $A'P$, and draw PK at right angles to $A'P$, meeting the transverse axis in K ; then

$$\text{since } PN^2 : AN \cdot A'N :: BC^2 : AC^2 \text{ (Prop. X.)}$$

$$\therefore A'N \cdot NK : AN \cdot A'N :: BC^2 : AC^2,$$

$$\text{or } NK : AN :: BC^2 : AC^2,$$

$$\therefore QP : AQ,$$

$\therefore NQ$ is parallel to PK .

\therefore also QH drawn at right angles to QN is parallel to $A'P$.

$$\text{Hence } AH : HA' :: AQ : QP.$$

30. Let SH be the given base; and P, Q the points of trisection of a segment described upon it.

It is evident that PQ is parallel to SH .

Bisect SH in X , and draw XM perpendicular to PQ , bisecting it in M ; then

$$SP = PQ = 2 PM.$$

\therefore the locus of P is a hyperbola whose focus is S , and directrix XM .

31. Since the triangles SCs , Tct are equal (*Prop. XXII.*)

\therefore also the triangles TVS , tVs are equal,

and the $\angle S VT$ is equal to the $\angle s Vt$,

$\therefore VS : Vs :: Vt : VT$ (*Euclid, VI. 14.*)

32. See fig. *Prop. XII.*

Let $S'Y'$ meet the auxiliary circle in Z' ; and draw $Z'M'$ perpendicular to $S'P$, then

$$S'M' : S'Z' :: S'Y' : S'P,$$

$$\begin{aligned}\therefore S'M' \cdot S'P &= S'Z' \cdot S'Y', \\ &= BC^2 \text{ (*Prop. XII.*)}\end{aligned}$$

$\therefore ZM'$ is the chord of contact of the pair of tangents drawn from P to the circle whose centre is S' , and radius BC .

Again, since CY produced passes through Z' , and CY is parallel to $S'P$,

\therefore the $\angle CZM'$ is a right angle,

$\therefore ZM'$ touches the auxiliary circle.

33. (1) Let C be between A and B ; and let the tangents AQ , BR meet in P ; then

$$\begin{aligned}AP \curvearrowright PB &= AQ \curvearrowright BR, \\ &= AC \curvearrowright BC.\end{aligned}$$

Therefore the locus of P is a hyperbola, of which A and B are the foci; unless AC is equal to BC , when the locus is a straight line bisecting AB at right angles.

(2) Let C be on AB produced, and let the tangents AQ , BR as before meet in P ; then,

$$\begin{aligned}AP + BP &= AQ + BR, \\ &= AC + BC,\end{aligned}$$

\therefore the locus of P is an ellipse.

34. Draw QM perpendicular to AA' ; then

$$\begin{aligned} QM : AM &:: PN : AN, \\ \text{and } QM : A'M &:: P'N : A'N, \\ \therefore QM^2 : AM \cdot A'M &:: PN^2 : AN \cdot A'N, \\ &:: BC^2 : AC^2. \end{aligned}$$

\therefore the locus of Q is a hyperbola, of which AC and BC are the semi-axes.

$$\begin{aligned} 35. \text{ Since } PF \cdot CD = AC \cdot BC, \quad \left. \begin{array}{l} \text{and } PF \cdot PG = BC^2. \end{array} \right\} & \text{(Prop. XXV. Cor.)} \\ \therefore CD : PG &:: AC : BC; \end{aligned}$$

hence, when the hyperbola is rectangular,

$$\begin{aligned} PG = CD = PL & \text{(Prop. XXIII.)} \\ \text{and } PL = Pl & \text{(Prop. XIX. Cor.)} \\ \therefore \text{ the angle } LGl & \text{ is a right angle.} \end{aligned}$$

36. Let A be the common vertex, and NPQ a common ordinate to the ellipse and parabola; then

$$PN^2 : AN \cdot A'N :: BC^2 : AC^2,$$

or denoting the diameter of the circle of curvature at A by $2R$.

$$\begin{aligned} PN^2 : AN \cdot A'N &:: R \cdot AC : AC^2 \text{ (Prop. XXVI. Ellipse)} \\ &:: R : AC, \\ &:: 2R \cdot AN : 2AC \cdot AN, \\ \therefore PN^2 : 2R \cdot AN &:: AN \cdot A'N : 2AC \cdot AN; \\ &:: A'N : 2AC. \end{aligned}$$

$$\begin{aligned} \text{But } QN^2 &= 4AS \cdot AN, \\ &= 2R \cdot AN \text{ (Prop. XIX. Parabola)} \end{aligned}$$

$$\therefore PN^2 : QN^2 :: A'N : 2AC,$$

Now in the ellipse $A'N$ is less than $2AC$.

$$\therefore PN \text{ is } < QN.$$

Again, if NP is an ordinate of the hyperbola, the same demonstration applies, except that

$$\begin{aligned} \bullet A'N & \text{ is } > 2AC, \\ \text{and } \therefore PN & > QN. \end{aligned}$$

37. Produce RQ to meet the asymptote in r , and draw PL parallel to CR' , meeting CR in L ; then

since Rr is bisected in Q , (Prop. XIX. Cor. 1)

$$\begin{aligned} \therefore 2QK & = CR, \\ & = CL + RL. \end{aligned}$$

$$\text{But } RL = P'H',$$

since $RP = R'P'$ (Prop. XIX.)

$$\begin{aligned} \therefore 2QK & = CL + P'H', \\ & = PH + P'H'. \end{aligned}$$

38. Draw PH, PH' , parallel to one asymptote, and $PK, P'K'$ parallel to the other.

Let $PH, P'K'$ meet in Q' , and $P'H', PK$ in Q ; then

$$PH \cdot PK = P'H' \cdot P'K' \text{ (Prop. XXI.)}$$

$$\begin{aligned} \therefore PH : P'K' & :: P'H' : PK, \\ \text{or } CK : KQ & :: CK' : K'Q', \\ \therefore QQ' & \text{ passes through } C. \end{aligned}$$

39. Draw PH, PK parallel to the sides CF, CE of the rectangle; then

$$4PH \cdot PK = CE \cdot CF,$$

and is therefore constant.

Hence the locus of P is a rectangular hyperbola, of which the sides of the rectangle are asymptotes.

40. Let the tangent at P meet the asymptotes CM, CN in L and l .

Join MN ; then

PM and PN are tangents to the ellipse (*Art. 36, Def.*)

\therefore the tangent at Q is parallel to MN (*Prop. XVIII. Ellipse.*)

Also, since LI is bisected in P (*Prop. XIX. Cor. 1*)

$\therefore CL$ and CI are bisected in M and N ,

$\therefore LI$ is parallel to MN ,

\therefore the tangents at P and Q are parallel.

$$41. \quad NP \cdot NQ = AN \cdot A'N,$$

$$\therefore PN^2 : NP \cdot NQ :: BC^2 : AC^2,$$

$$\text{or } PN : QN :: BC^2 : AC^2,$$

$$:: BC : B'C,$$

if $B'C$ be a third proportional to BC and AC ,

$$\therefore QN^2 : PN^2 :: B'C^2 : BC^2,$$

$$\text{and } PN^2 : AN \cdot A'N :: BC^2 : AC^2,$$

$$\therefore QN^2 : AN \cdot A'N :: B'C^2 : AC^2,$$

\therefore the locus of Q is a hyperbola, of which AC and $B'C$ are the transverse and conjugate semi-axes.

42. See fig. *Prop. XXV.*

Let PP' be the diameter, and Q any point on the curve.

Join PQ , $P'Q$, and draw CU parallel to $P'Q$, bisecting PQ in the point U .

Then if CU' be drawn parallel to PQ , CU and CU' will be conjugate diameters, and therefore (see *Prob. 15*) will make equal angles with CR , as also with CP and CD .

$$\begin{aligned} \text{Hence } \angle QPP' - \angle QP'P &= \angle U'CP' - \angle UCP, \\ &= \angle U'CP' - \angle U'CD, \\ &= \angle P'CD. \end{aligned}$$

43. Let Tpt be a common tangent to the hyperbola and any one of the elliptic quadrants APB , and let it meet the asymptotes CA , CB in T and t .

Draw the ordinates PN , Pn parallel respectively to Ct and CT ; then

since TPt is bisected in P (*Prop. XIX. Cor. 1*)

$\therefore CT$ is bisected in N .

But from the ellipse $AC^2 = CN \cdot CT$,

$$\therefore AC^2 = 2CN^2.$$

So also

$$BC^2 = 2Cn^2,$$

$$\therefore AC^2 : BC^2 :: CN^2 : Cn^2,$$

$$\therefore AC : BC :: CN : Cn,$$

$$\therefore AC \cdot BC : BC^2 :: CN \cdot Cn : Cn^2,$$

$$\begin{aligned}\therefore AC \cdot BC &= 2CN \cdot Cn, \\ &= 2CS^2. \quad (\text{Prop. XXI.})\end{aligned}$$

44. Produce OP to meet CD , the diameter conjugate to CP , in F ; then

$$OP : PQ :: CP : PF,$$

$$\begin{aligned}\therefore OP \cdot PF &= PQ \cdot CP, \\ &= CP^2,\end{aligned}$$

$\therefore O$ is the centre of the circle of curvature at P . (*Prop. XXIX.*)

45. See fig. *Prop. XXV.*

Let Ct and CL be conjugate diameters of an ellipse, with C as centre, described touching the hyperbola, of which Ct and CL are asymptotes, in P .

Draw the common tangent LPt , and from the point L draw LDl' , touching the conjugate hyperbola in D ; then

CP and CD are conjugate diameters of the hyperbola,
and PD is parallel to ll' . (*Prop. XXIII.*)

Join PD , meeting CL in M ; then
 since PD is a parallelogram,
 $\therefore MD = PM$.

Hence, since CL and CL' are conjugate diameters of the ellipse,

$\therefore D$ is a point on the ellipse;

and, since the tangent to the ellipse at P is parallel to CD ,

\therefore the tangent to the ellipse at D is parallel to CP ,

$\therefore LD$ touches the ellipse at D ,

or the ellipse and hyperbola have a common tangent at D ,
 and therefore touch one another.

Also CP and CD are conjugate diameters of the ellipse, as well as of the hyperbola.

46. Let Q, Q' be the points where the common tangents $QP, Q'P'$ touch the hyperbola, and P, P' the points where they touch the ellipse.

Through O , the point of intersection of QP and $Q'P'$, draw CO , meeting PP' and QQ' in U and V respectively.

Join CQ, CQ' meeting the ellipse in q, q' ; then
 since QQ', PP' are bisected in V and U , they are evidently parallel.

Now, since a line drawn through q , parallel to PP' and QQ' , will be bisected by CUV , whether it be terminated by the ellipse or CQ' , it is evident that

qq' is parallel to QQ' .

Hence $CQ : CQ' :: Cq : Cq'$.

\therefore an ellipse similar to the given ellipse, and having also C for its centre, can be described through the points Q and Q' .

47. Let $PQ, P'O'$ be the radii of curvature at P and P' respectively; and let $PF, P'F'$ be the perpendiculars from P and P' upon the diameters CD and CD' , conjugate respectively to CP and CP' ; then

$$\begin{aligned}
 PQ \cdot PF &= CD^2 \text{ (Prop. XXIX.)} \\
 \therefore PO \cdot PF \cdot CD &= CD^3, \\
 \text{or } PO \cdot AC \cdot BC &= CD^3 \text{ (Prop. XXV. Cor.)} \\
 \text{So also } P'O' \cdot AC \cdot BC &= CD'^3, \\
 \therefore PO : P'O' &:: CD^3 : CD'^3, \\
 &:: CP^3 : CP'^3 \text{ (Prob. 16),}
 \end{aligned}$$

since the hyperbola is rectangular.

48. Let R and S be two of the points of intersection on the same side of PQ ; and let PQ intersect the hyperbola in p and q .

Draw OV bisecting PQ in V ; then

$$\text{since } Pp = Qq \text{ (Prop. XIX.)}$$

$\therefore V$ is also the middle point of pq .

Now, since the chords which are drawn parallel to PQ are bisected both in the ellipse and hyperbola by OV , it is evident that the chord drawn from R , parallel to PQ , must meet the ellipse and hyperbola in the same point.

Hence RS is parallel to PQ .

So also, if R' , S' be the points of intersection of the hyperbola and ellipse on the other side of PQ ,

$R'S'$ is parallel to PQ .

Produce PR to meet OV in T , and produce TS to meet PQ in Q' ; then if RS meet OV in M ,

$$\begin{aligned}
 Q'V : TV &:: SM : TM, \\
 &:: RM : TM, \\
 &:: PV : TV, \\
 &:: QV : TV,
 \end{aligned}$$

$\therefore Q$ and Q' coincide,

and TS passes through the point Q .

Hence, since the tangents to the hyperbola at R and S meet OV in the same point (Prop. XXVII. Cor.), and PR and

QS have also been proved to intersect on OV , it is evident that,

if PR is the tangent at R ,

QS must also be the tangent at S .

Lastly, whether U be the point where RQ or PS intersects OV ,

$$MU : UV :: RM : PV,$$

$\therefore RQ$ and PS meet on the line OV , which bisects PQ .

49. Let the points A and B be on the same branch of the hyperbola.

Let AQ and BP intersect in R , join CR , and produce CR to meet AB in O ; then, since the angles at P and Q are right angles, and the three perpendiculars from the angles of a triangle on the opposite sides meet in a point,

$\therefore CR$ is at right angles to AB .

Now by similar triangles

$$CO : OA :: BO : OR,$$

$$\therefore CO \cdot OR = OA \cdot BO.$$

Hence, by the converse of the *Cor.* of *Prop.* XXVIII., since the hyperbola is rectangular, C is also a point on the hyperbola, but situated on the opposite branch to that on which AB is drawn, and on which C moves.

PROBLEMS ON THE SECTIONS OF THE CONE.

1. See fig. *Prop.* I.

Let C be the centre of the sphere inscribed in the cone, and touching the plane of the parabolic section in the focus S .

Join CA , CE ; then

since AS is parallel to Oe ,

it is evident that

the $\angle ACO$ is a right angle

Now $SA : AO :: AE : AO$.

But since $AE : AC :: AC : AO$,

$\therefore AE : AO :: AC^2 : AO^2$.

Hence $SA : AO :: AC^2 : AO^2$,

\therefore the ratio of SA to AO is independent of the position of the point A , and depends only upon the vertical angle of the cone.

The foci therefore of all parabolic sections made by planes perpendicular to the plane of the paper, will be upon a straight line drawn through O ; and the foci of all the parabolic sections that can be made by any planes, upon the surface of the cone formed by the revolution of this line round the axis of the given cone.

2. See fig. 2, *Prop.* I.

$CS : CA :: SA : AX$,

$:: AE : AX$.

Now if the ratio of AE to AX is fixed,

since AEX is a fixed angle,

and AE is less than AX ,

and AXE is therefore less than a right angle;

the $\angle AXE$ is fixed, and can have only one value (*Euclid*, VI. 7.)

\therefore also the $\angle SAO$ is fixed.

Draw AC to the centre of the inscribed sphere bisecting the angle SAO ; then the angle CAE is a fixed angle.

Now $SA : AO :: AE : AC$,

and the ratio of AB to AO is compounded of the ratios of $AE : AC$ and $AC : AO$; but since the $\angle CAE$ is fixed, the ratios of $AE : AC$, and $AC : AO$ are both fixed ratios.

Hence $SA : AO$ is a fixed ratio.

Therefore, as in the preceding problem, the locus of S is a cone, having O for its vertex.

In the same manner it may be shown that the other focus S' will always lie on another cone which has also O for its vertex.

3. See fig. 2, *Prop. I.*

Let AA' be the major axis of one of the ellipses.

Since the cutting planes are parallel, the major axes of the elliptic sections will be parallel also.

Hence, if O be joined with the middle point of AA' , the line so drawn will also pass through the centres of all the other elliptic sections.

If through this line a plane be drawn perpendicular to the plane of the paper, the extremities of the minor axes will all lie on the two lines in which this plane intersects the surface of the cone.

4. See fig. 1, *Prop. I.*

The latus rectum $= 4AS$,

and in Prob. 1 it has been shown that AS bears a ratio to AO , depending only upon the vertical angle of the cone.

Hence the latus rectum varies as AO .

5. See fig. 3, *Prop.* I.

The angle KOL represents half the angle between the asymptotes of the section made by a plane parallel to KOL .

Now, in the rectangular hyperbola, the angle between the asymptotes is a right angle, and the greatest value which KOL admits of is half the vertical angle of the cone.

Hence, if the vertical angle of the cone is less than a right angle, it will be impossible to cut it in such a manner that the section may be an equilateral hyperbola.

When the vertical angle is a right angle any section made by a plane parallel to a plane through the axis will intersect the curve in a rectangular hyperbola.

When the angle ROr is greater than a right angle, bisect OR in V , and draw VZ at right angles to OR , making VZ equal to OV , and join OZ ; then

$$OR^2 = 2 OZ^2.$$

With centre O , and radius OZ , describe a circle, cutting Rr in L ; then

$$\begin{aligned} KL^2 &= OK^2 - OL^2, \\ &= OR^2 - OL^2, \\ &= 2 OL^2 - OL^2, \\ &= OL^2. \end{aligned}$$

Hence the angle KOL is half a right angle, and therefore any plane parallel to KOL will intersect the cone in an equilateral hyperbola.

6. When the cutting plane is parallel to OI and Oi , the perpendicular from O on the cutting plane is the conjugate semi-axis of both hyperbolas.

The latera recta are therefore inversely proportional to the transverse axes. (Art. 48.)

Also OD , Od , and OD , Od are parallel to the asymptotes of the hyperbolas in the cones whose axes are respectively OI and Oi .

Let Oa , ab , measured along and perpendicular to OI , represent the semi-axes of the one hyperbola, and Oa' , $a'b'$,

measured along and perpendicular to Oi , the semi-axes of the other hyperbola.

$$\begin{aligned} \text{Then} \quad & Oa : ab :: OI : Oi, \\ \text{and} \quad & a'b' : Oa' :: OI : Oi; \\ \text{since} \quad & ab = a'b', \\ & Oa : Oa' :: OI^2 : Oi^2. \end{aligned}$$

\therefore the latera recta are as $Oi^2 : OI^2$.

Next, when the cutting plane is perpendicular to OI and Oi , the transverse axis of the hyperbola is equal to the major axis of the ellipse.

Let AA' be the common axis (see fig. 3, *Prop. I.*), C the common centre; then, if Cuu' be drawn parallel to OI , meeting OD , Od' in u and u' ; then

the latera recta are as $CU \cdot CU' : Cu \cdot Cu'$ (*Prop. I.*)
as $Oi^2 : OI^2$.

APPENDIX.

1. THE following is an independent solution of *Prob. 26* of the Parabola.

Join AP , AQ , and let them meet in p and q , the line drawn through O at right angles to the axis; then since

$$Op : PM :: AO : AM,$$

$$\text{and } PM : 4AS :: AM : PM,$$

$$\therefore Op : 4AS :: AO : PM.$$

$$\text{So also } Oq : 4AS :: AO : QN,$$

$$\therefore Op : Oq :: QN : PM,$$

$$:: QO : PO,$$

$$\therefore pQ \text{ is parallel to } Pq,$$

$$\therefore Ap : AP :: Aq : Aq,$$

$$\therefore AO : AM :: AM : AO,$$

$$\therefore AO^2 = AM \cdot AN.$$

2. The following proof of the more general property, of which *Prob. 44* of the ellipse is a particular case, is deserving of notice.

Let O be any point. It is required to find the locus of the middle points of chords drawn through O

Join OC , and produce it to meet the ellipse in K and K' .

Draw any chord POp , and let Q be its middle point, and through K draw the chord KR parallel to POp . Produce CQ to meet KR in r ; then KR is bisected in r .

$$\text{Hence } OQ : Kr :: OC : CK,$$

$$\therefore OQ : KR :: OC : KK',$$

$$\text{i.e. } OQ \text{ bears a constant ratio to } KR,$$

\therefore the locus of Q is an ellipse similar to the given ellipse.

LONDON
R. CLAY, SON, AND TAYLOR, PRINTERS,
BREAD STREET HILL.

Cambridge,
And 23, HENRIETTA STREET, COVENT GARDEN,
London, W.C.

MACMILLAN AND CO.'S
DESCRIPTIVE CATALOGUE OF
CAMBRIDGE CLASS BOOKS.

This Series of CAMBRIDGE CLASS-BOOKS FOR THE USE OF SCHOOLS AND COLLEGES, which have been issued at intervals during the last ten years, is intended to embrace all branches of Education, from the most elementary to the most advanced, and to keep pace with the latest discoveries in Science.

Of those hitherto published the sale of many thousands is a sufficient indication of the manner in which they have been appreciated by the public.

A SERIES of a more Elementary character is in preparation, a list of which will be found at the back of this Catalogue.

CAMBRIDGE CLASS BOOKS

FOR

SCHOOLS AND COLLEGES.

WORKS by the Rev. BARNARD SMITH, M.A.

Fellow of St. Peter's College Cambridge.

Arithmetic and Algebra

In their Principles and Applications.

With numerous Examples, systematically arranged.

Eighth Edition. 696 pp. (1861). Crown 8vo. strongly bound in cloth. 10s. 6d.

The first edition of this work was published in 1854. It was primarily intended for the use of students at the Universities, and for Schools which prepare for the Universities. It has however been found to meet the requirements of a much larger class, and is now extensively used in *Schools and Colleges both at home and in the Colonies*. It has also been found of great service for students preparing for the MIDDLE-CLASS and CIVIL AND MILITARY SERVICE EXAMINATIONS, from the care that has been taken to elucidate the principles of all the Rules. Testimony of its excellence has been borne by some of the highest practical and theoretical authorities; of which the following from the late DEAN PEACOCK may be taken as a specimen.

"Mr. Smith's Work is a most useful publication. The Rules are stated with great clearness. The Examples are well selected and worked out with just sufficient detail without being encumbered by too minute explanations; and there prevails throughout it that just proportion of theory and practice, which is the crowning excellence of an elementary work."

2. Arithmetic

For the Use of Schools.

New Edition (1862) 348 pp.
Crown 8vo. strongly bound in cloth,
4s. 6d. Answers to all the Questions.

3. Key to the above, containing
Solutions to all the Questions in the
latest Edition. Crown 8vo. cloth.
392 pp. Second Edition. 8s. 6d.

To meet a widely expressed wish, the ARITHMETIC was published separately from the larger work in 1854, with so much alteration as was necessary to make it quite independent of the ALGEBRA. It has now a large and increasing sale in all classes of Schools at home and in the Colonies. A very copious collection of Examples, under each rule, has been embodied in the work in a systematic order, and a Collection of Miscellaneous Papers in all branches of Arithmetic have been appended to the book.

4. Exercises in Arithmetic. 104 pp. Crown 8vo. (1860)
2s. Or with ANSWERS, 2s. 6d. Also
sold separately in 2 Parts 1s. each.
Answers, 6d.

The EXERCISES have been published in order to give the pupil examples of every rule in Arithmetic, and a great number have been carefully compiled from the latest University and School Examination Papers.

WORKS by ISAAC TODHUNTER, M.A. F.R.S.

Fellow and Principal Mathematical Lecturer of St. John's College, Cambridge.

1. Algebra.

For the Use of Colleges and Schools.

Third Edition. 542 pp. (1862).

Strongly bound in cloth. 7s. 6d.

This work contains all the propositions which are usually included in elementary treatises on Algebra, and a large number of *Examples for Exercise*. The author has sought to render the work easily intelligible to students without impairing the accuracy of the demonstrations, or contracting the limits of the subject. The Examples have been selected with a view to illustrate every part of the subject, and as the number of them is about sixteen hundred and fifty, it is hoped they will supply ample exercise for the student. Each set of Examples has been carefully arranged, commencing with very simple exercises, and proceeding gradually to those which are less obvious.

2. Plane Trigonometry.

For Schools and Colleges.

Second Edit. 279 pp. (1860). Crn. 8vo.

Strongly bound in cloth. 5s.

The design of this work has been to render the subject intelligible to beginners, and at the same time to afford the student the opportunity of obtaining all the information which he will require on this branch of Mathematics. Each chapter is followed by a set of Examples; those which are entitled *Miscellaneous Examples*, together with a few in some of the other sets, may be advantageously reserved by the student for exercise after he has made some progress in the subject. As the Text and Examples of the present work have been tested by considerable experience in teaching, the hope is entertained that they will be suitable for imparting a sound and comprehensive knowledge of Plane Trigonometry, together with readiness in the application of this knowledge to the solution of problems. In the Second Edition the hints for the solution of the Examples have been considerably increased.

Spherical Trigonometry

For the Use of Colleges and Schools.

112 pp. Crown 8vo. (1869).

Strongly bound in cloth. 4s. 6d.

This work is constructed on the same plan as the *Treatise on Plane Trigonometry*, to which it is intended as a sequel. Considerable labour has been expended on the text in order to render it comprehensive and accurate, and the Examples, which have been chiefly selected from University and College Papers, have all been carefully verified.

4. The Integral Calculus

And its Applications.

With numerous Examples.

Second Edition. 342 pp. (1862).

Crown 8vo. cloth. 10s. 6d.

In writing the present *Treatise on the Integral Calculus*, the object has been to produce a work at once elementary and complete—adapted for the use of beginners, and sufficient for the wants of advanced students. In the selection of the propositions, and in the mode of establishing them, the author has endeavoured to exhibit fully and clearly the principles of the subject, and to illustrate all their most important results. In order that the student may find in the volume all that he requires, a large collection of Examples for exercise has been appended to the different chapters.

5. The

Differential Calculus.

With numerous Examples.

Third Edition, 398 pp. (1860).

Crown 8vo. cloth, 10s. 6d.

This work is intended to exhibit a comprehensive view of the Differential Calculus on the method of Limits. In the more elementary portions, explanations have been given in considerable detail, with the hope that a reader who is without the assistance of a tutor may be enabled to acquire a competent acquaintance with the subject. More than one investigation of a theorem has been frequently given, because it is believed that the student derives advantage from viewing the same proposition under different aspects, and that in order to succeed in the examinations which he may have to undergo, he should be prepared for a considerable variety in the order of arranging the several branches of the subject, and for a corresponding variety in the mode of demonstration.

FOR SCHOOLS AND COLLEGES.

WORKS by ISAAC TOPHUNTER, M.A., F.R.S.—continued.

6. Analytical Statics.

With numerous Examples.

Second Edition. 330 pp. (1858).

Crown 8vo. cloth. 10s. 6d.

In this work will be found all the propositions which usually appear in treatises on Theoretical Statics. To the different chapters Examples are appended, which have been selected principally from the University and College Examination Papers; these will furnish ample exercise in the application of the principles of the subject.

7. AN ELEMENTARY TREATISE ON THE

Theory of Equations.

With a Collection of Examples.

Crown 8vo. cloth. 7s. 6d.

This treatise contains all the propositions which are usually included in elementary treatises on the Theory of Equations, together with a collection of Examples for exercise. It may be read by those who are familiar with Algebra, since no higher knowledge is assumed, except in Arts. 175, 267, 308—314, which may be postponed by those who are not acquainted with De Moivre's Theorem in Trigonometry. This work may in fact be regarded as a sequel to that on Algebra by the same writer, and accordingly the student has occasionally been referred to the treatise on Algebra for preliminary information on some topics here discussed.

8. EXAMPLES OF

Analytical Geometry of Three Dimensions.

76 pp. (1858). Crown 8vo. cloth. 4s.

A collection of examples in illustration of Analytical Geometry of Three Dimensions has long been required both by students and teachers, and the present work is published with the view of supplying the want.

9. Plane Co-Ordinate Geometry

AS APPLIED TO THE STRAIGHT LINE
AND THE CONIC SECTIONS;

With numerous Examples.

Third and Cheaper Edition.

Crown 8vo. cl. 326 pp. (1862). 7s. 6d.

This Treatise exhibits the subject in a simple manner for the benefit of beginners, and at the same time includes in one volume all that students usually require. The Examples at the end of each chapter will, it is hoped, furnish sufficient exercise, as they have been carefully selected with the view of illustrating the most important points, and have been tested by repeated experience with pupils. In consequence of the demand for the work proving much greater than had been originally anticipated, a large number of copies of the Third Edition has been printed, and a considerable reduction effected in the price.

10. History of the Progress

OF THE

Calculus of Variations

During the Nineteenth Century.

8vo. cloth. 532 pp. (1861). 12s.

It is of importance that those who wish to cultivate any subject may be able to ascertain what results have already been obtained, and thus reserve their strength for difficulties which have not yet been conquered. And those who merely desire to ascertain the present state of a subject without any purpose of original investigation will often find that the study of the past history of that subject assists them materially in obtaining a sound and extensive knowledge of the condition which it has attained. The Author has endeavoured in this work to ascertain distinctly what has been effected in the Progress of the Calculus, and to form some estimate of the manner in which it has been effected: accordingly, unless the contrary is distinctly stated, it may be assumed that any treatise or memoir relating to the Calculus of Variations which is described in this work has undergone thorough examination and study.

By J. H. PRATT, M.A.

Archdeacon of Calcutta, late Fellow of
Gonville and Caius College, Cambridge.**A Treatise on
Attractions,***La Place's Functions, and the Figure
of the Earth.*Second Edition. Crown 8vo. 126 pp.
(1861). cloth. 6s. 6d.

In the present Treatise the author has endeavoured to supply the want of a work on a subject of great importance and high interest—La Place's Coefficients and Functions and the calculation of the Figure of the Earth by means of his remarkable analysis. No student of the higher branches of Physical Astronomy should be ignorant of La Place's analysis and its result—"a calculus," says Airy, "the most singular in its nature and the most powerful in its application that has ever appeared."

By P. G. TAIT, M.A., and
W. J. STEELE, B.A.

Late Fellows of St. Peter's Coll. Cambridge.

Dynamics of a Particle.*With numerous Examples.*

304 pp. (1856). Cr. 8vo. cl. 10s. 6d.

In this Treatise will be found all the ordinary propositions connected with the Dynamics of Particles which can be conveniently deduced without the use of D'Alembert's Principles. Throughout the book will be found a number of illustrative Examples introduced in the text, and for the most part completely worked out; others, with occasional solutions or hints to assist the student are appended to each Chapter.

By the Rev. G. F. CHILDE, M.A.
Mathematical Professor in the South African
College.**Singular Properties of
the Ellipsoid***And Associated Surfaces of the n th
Degree.*

152 pp. (1861). 8vo. boards. 10s. 6d.

As the title of this volume indicates, its object is to develop peculiarities in the Ellipsoid; and further, to establish analogous properties in unlimited congenetic series of which this remarkable surface is a constituent.

By Rev. G. PARKINSON, B.D.

Fellow and Praelector of St. John's Coll.
Cambridge.**1. Elementary Treatise
on Mechanics.***With a Collection of Examples.*Second Edition. 345 pp. (1861).
Crown 8vo. cloth. 9s. 6d.

The Author has endeavoured to render the present volume suitable as a Manual for the junior classes in Universities and the higher classes in Schools. With this object there have been included in it those portions of theoretical Mechanics which can be conveniently investigated without the Differential Calculus, and with one or two short exceptions the student is not presumed to require a knowledge of any branches of Mathematics beyond the elements of Algebra, Geometry, and Trigonometry. A collection of Problems and Examples has been added, chiefly taken from the Senate-House and College Examination Papers—which will, it is trusted, be found useful as an exercise for the student. In the Second Edition several additional propositions have been incorporated in the work for the purpose of rendering it more complete, and the Collection of Examples and Problems has been largely increased.

2. A Treatise on Optics.

304 pp. (1859). Crown 8vo. 10s. 6d.

The present work may be regarded as a new edition of the *Treatise on Optics*, by the Rev. W. N. Griffin, which being some time ago out of print, was very kindly and liberally placed at the disposal of the author. The author has freely used the liberty accorded to him, and has rearranged the matter with considerable alterations and additions—especially in those parts which required more copious explanation and illustration to render the work suitable for the present course of reading in the University. A collection of Examples and Problems has been appended, which are sufficiently numerous and varied in character to afford an useful exercise for the student: for the greater part of them recourse has been had to the Examination Papers set in the University and the several Colleges during the last twenty years.

Subjoined to the copious Table of Contents the author has ventured to indicate an elementary course of reading not unsuitable for the requirements of the First Three Days in the Cambridge Senate House Examinations.

FOR SCHOOLS AND COLLEGES.

By J. B. PHEAR, M.A.

Fellow and late Mathematical Lecturer of
Clare College.

Elementary Hydrostatics

With numerous Examples and Solutions.

Second Edition. 166 pp. (1857).

Crown 8vo. cloth. 6s. 6d.

"An excellent Introductory Book. The definitions are very clear; the descriptions and explanations are sufficiently full and intelligible; the investigations are simple and scientific. The examples greatly enhance its value."—*ENGLISH JOURNAL OF EDUCATION*.

This Edition contains 147 Examples, and solutions to all these examples are given at the end of the book.

By G. B. AIRY, M.A.

Astronomer Royal.

1. Mathematical Tracts.

*On the Lunar and Planetary Theories,
Figure of the Earth, the Undulatory
Theory of Optics, &c.*

Fourth Edition. 400 pp. (1858).

8vo. 15s.

2. Theory of Errors of Observations

And the Combination of Observations.

103 pp. (1861). Crown 8vo. 6s. 6d.

In order to spare astronomers and observers in natural philosophy the confusion and loss of time which are produced by referring to the ordinary treatises embracing both branches of Probabilities, the author has thought it desirable to draw up this work, relating only to Errors of Observation, and to the rules derivable from the consideration of these Errors, for the Combination of the Results of Observations. The Author has thus also the advantage of entering somewhat more fully into several points of interest to the observer, than can possibly be done in a General Theory of Probabilities.

By GEORGE BOOLE, D.C.L.

Professor of Mathematics in the Queen's
University, Ireland.

1. Differential Equations

468 pp. (1859). Crown 8vo. cloth. 14s.

The author has endeavoured in this treatise to convey as complete an account of the present state of knowledge on the subject of Differential Equations as was consistent with the idea of a work intended, primarily, for elementary instruction. The object has been first of all to meet the wants of those who had no previous acquaintance with the subject, and also not quite to disappoint others who might seek for more advanced information. The earlier sections of each chapter contain that kind of matter which has usually been thought suitable for the beginner, while the latter ones are devoted either to an account of recent discovery, or to the discussion of such deeper questions of principle as are likely to present themselves to the reflective student in connection with the methods and processes of his previous course.

2. The Calculus of Finite Differences.

248 pp. (1860). Crown 8vo. cloth.
10s. 6d.

In this work particular attention has been paid to the connexion of the methods with those of the Differential Calculus—a connexion which in some instances involves far more than a merely formal analogy. The work is in some measure designed as a sequel to the Author's *Treatise on Differential Equations*, and it has been composed on the same plan.

By J. C. SNOWBALL, M.A.

Late Fellow of St. John's Coll. Cambridge.

Plane and Spherical Trigonometry.

*With the Construction and Use of Tables
of Logarithms.*

Ninth Edition. 240 pp. (1857). Crown
8vo. 7s. 6d.

In preparing a new edition, the proofs of some of the more important propositions have been rendered more strict and general; and a considerable addition, of more than two hundred Examples, taken principally from the questions in the Examinations of Colleges and the University, has been made to the collection of Examples and Problems for practice.

By **W. H. DREW, M.A.**

Second Master of Blackheath School.

**Geometrical Treatise on
Conic Sections.***With a copious Collection of Examples.*

Second Edition. Crown 8vo. cloth.

4s. 6d.

In this work the subject of Conic Sections has been placed before the student in such a form that, it is hoped, after mastering the elements of Euclid, he may find it an easy and interesting continuation of his geometrical studies. With a view also of rendering the work a complete Manual of what is required at the Universities, there have been either embodied into the text, or inserted among the examples, every book-work question, problem, and rider, which has been proposed in the Cambridge examinations up to the present time.

Solutions to the Problems in Drew's Conic Sections.

Crown 8vo. cloth. 4s. 6d.

By **G. H. PUCKLE, M.A.**

Principal of Windermere College.

**Conic Sections and
Algebraic Geometry.***With numerous Easy Examples Progressively arranged.*

Second Edition. 264 pp. (1856).

Crown 8vo. 7s. 6d.

This book has been written with special reference to those difficulties and misapprehensions which commonly beset the student when he commences. With this object in view, the earlier part of the subject has been dwelt on at length, and geometrical and numerical illustrations of the analysis have been introduced. The Examples appended to each section are mostly of an elementary description. The work will, it is hoped, be found to contain all that is required by the upper classes of schools and by the generality of students at the Universities.

By **EDWARD JOHN ROUTH, M.A.**

Fellow and Assistant Tutor of St. Peter's College, Cambridge.

**Dynamics of a System
of Rigid Bodies.***With numerous Examples.*336 pp. (1860). Crown 8vo. cloth.
10s. 6d.

CONTENTS: Chap. I. Of Moments of Inertia.—II. D'Alembert's Principle.—III. Motion about a Fixed Axis.—IV. Motion in Two Dimensions.—V. Motion of a Rigid Body in Three Dimensions.—VI. Motion of a Flexible String.—VII. Motion of a System of Rigid Bodies.—VIII. Of Impulsive Forces.—IX. Miscellaneous Examples.

The numerous Examples which will be found at the end of each chapter have been chiefly selected from the Examination Papers set in the University and Colleges of Cambridge during the last few years.

By **R. D. BEASLEY, M.A.**

Head Master of Grantham School.

Plane Trigonometry.

AN ELEMENTARY TREATISE.

With a numerous Collection of Examples.

106 pp. (1858), strongly bound in cloth, 3s. 6d.

This Treatise is specially intended for use in Schools. The choice of matter has been chiefly guided by the requirements of the three days' Examination at Cambridge, with the exception of proportional parts in logarithms, which have been omitted. About four hundred Examples have been added, mainly collected from the Examination Papers of the last ten years, and great pains have been taken to exclude from the body of the work any which might dishearten a beginner by their difficulty.

The
Cambridge Year Book
AND UNIVERSITY ALMANACK
For 1862.

Crown 8vo. 228 pp. price 2s. 6d.

The specific features of this annual publication will be obvious at a glance, and its value to teachers engaged in preparing students for, and to parents who are sending their sons to, the University, and to the public generally, will be clear.

1. The whole mode of proceeding in entering a student at the University and at any particular College is stated.

2. The course of the studies as regulated by the University examinations, the manner of these examinations, and the specific subjects and times for the year 1862, are given.

3. A complete account of all Scholarships and Exhibitions at the several Colleges, their value, and the means by which they are gained.

4. A brief summary of all Grades of the Senate, Degrees conferred during the year 1861, and University news generally are given.

5. The Regulations for the LOCAL EXAMINATION of those who are not members of the University, to be held this year, with the names of the books on which the Examination will be based, and the date on which the Examination will be held.

* Published annually in January.

**Cambridge University
Examination Papers,**

Crown 8vo. 184 pp. 2s. 6d.

A Collection of all the Papers set at the Examinations for the Degrees, the various Triposes, and the Theological Certificates in the University, with List of Candidates Examined and of those Approved, and an Index to the Subjects. 1860-61.

Previous Examination, 1860—Previous Examination, 1861—B.A. Degree Examination, 1860—B.A. Degree Examination, Jan. and May, 1861—Bachelor of Laws Examination, 1860—Bachelor of Laws Examination, 1861—Bachelor of Medicine Examinations, 1861—Classical Tripos, 1861—Moral Sciences Tripos, 1861—Natural Sciences Tripos, 1861—Smith's Prizes, 1861—Chancellor's Medals for Legal and Classical Studies, 1861.

**Arithmetic in Theory
and Practice.**

For Advanced Pupils.

PART I. Crown 8vo. cloth. 3s. 6d.

This work forms the first part of a Treatise on Arithmetic, in which the Author has endeavoured, from very simple principles, to explain in a full and satisfactory manner all the important processes in that subject.

The proofs have in all cases been given in a form entirely arithmetical: for the author does not think that recourse ought to be had to Algebra until the arithmetical proof has become hopelessly long and perplexing.

At the end of every chapter several examples have been worked out at length, in which the best practical methods of operation have been carefully pointed out.

By N. M. FERRERS, M.A.

Fellow and Mathematical Lecturer of Gonville and Caius College, Cambridge.

**AN ELEMENTARY TREATISE ON
Trilinear Co-Ordinates.**

The Method of Reciprocal Polars, and the Theory of Projectiles.

154 pp. (1861). Cr. 8vo. cl. 6s. 6d.

The object of the Author in writing on this subject has mainly been to place it on a basis altogether independent of the ordinary Cartesian System, instead of regarding it as only a special form of abridged Notation. A short chapter on Determinants has been introduced.

Senate-House Mathematical Problems.

With Solutions.

1848-51. By FERRERS and JACKSON. 8vo. 15s. 6d.

1848-51. (RIDERS). By JAMESON. 8vo. 7s. 6d.

1854. By WALTON and MACKENZIE. 8vo. 10s. 6d.

1857. By CAMPION and WALTON. 8vo. 8s. 6d.

1860. By ROUTH and WATSON. Crown 8vo. 7s. 6d.

The above books contain Problems and Examples which have been set in the Cambridge Senate-house Examinations at various periods during the last twelve years, together with Solutions of the same. The Solutions are in all cases given by the Examiners themselves or under their sanction.

By C. W. UNDERWOOD, M.A.
Vice-Principal of the College Institution,
Liverpool.

A Short Manual of Arithmetic.

Fcp. 8vo. 96 pp. (1866). Limp cloth.
2s. 6d.

The object aimed at by the Compiler of this *Manual* is to bring before junior students so much of the Theory of Arithmetic as may be fairly expected of them, and to present it in such a form that the study of the Science may become to some extent a *mental training*. It is rather a *Grammar* of Arithmetic than a treatise on that subject, and should for the most part be committed to memory. It will be found well adapted for *ried voce* examination, and enable candidates to prepare themselves for the Local University Examinations. The Definitions are briefly and carefully worded. Each rule is stated so as to include the proof of it where this was possible.

By H. A. MORGAN, M.A.
Fellow of Jesus College, Cambridge.

A Collection of Mathematical Problems and Examples.

With Answers.

190 pp. (1858). Crown 8vo. 6s. 6d.

• This book contains a number of problems, chiefly elementary, in the Mathematical subjects usually read at Cambridge. They have been selected from the papers set during late years at Jesus College. Very few of them are to be met with in other collections, and by far the larger number are due to some of the most distinguished Mathematicians in the University.

Elementary Statics.

By the Rev. GEORGE RAWLINSON,
Professor of Applied Sciences, Elphinstone
Coll., Bombay.

Edited by the Rev. E. STURGES, M.A.

Rector of Kencott, Oxfordshire.

(150 pp.) 1860. Crown 8vo. cloth.
4s. 6d.

This work is published under the authority of H. M. Secretary of State for India for use in the Government Schools and Colleges in India.

By J. HERBERT LATHAM, M.A.
Civil Engineer.

THE CONSTRUCTION OF Wrought-Iron Bridges.

*Embracing the Practical Application
of the Principles of Mechanics to
Wrought-Iron Girder-Work.*

"The great merit of this book is that it deals with practice more than theory. All the calculations in the book connected with the strength of girders are based upon their actual application which abounds in practical investigations into girder-work in all its bearings, and will be welcomed as one of the most valuable contributions yet made to this important branch of engineering."—
ATHENÆUM.

By JOHN E. B. MAYOR, M.A.
Fellow and Classical Lecturer of St. John's
College, Cambridge.

1. Juvenal.

With English Notes.

464 pp. (1854). Crown 8vo. cloth.
10s. 6d.

"A School edition of Juvenal, which, for really ripe scholarship, extensive acquaintance with Latin literature, and familiar knowledge with Continental criticism, ancient and modern, is unsurpassed, we do not say among English School-books, but among English editions generally."—EDINBURGH REVIEW.

2. Cicero's Second Philippic.

With English Notes.

168 pp. (1861). Fcp. 8vo. cloth. 5s.

The Text is that of Halm's 2nd edition, (Leipzig, Weidmann, 1858), with some corrections from Madvig's 4th Edition (Copenhagen, 1858). Halm's Introduction has been closely translated, with some additions. His notes have been curtailed, omitted, or enlarged, at discretion; passages to which he gives a bare reference, are for the most part printed at length; for the Greek extracts an English version has been substituted. A large body of notes, chiefly grammatical and historical, has been added from various sources. A list of books useful to the student of Cicero, a copious Argument, and an Index to the introduction and notes, complete the book.

FOR SCHOOLS AND COLLEGES.

11

By C. MERIVALE, B.D.

Author of "A History of Rome," &c.

Sallust.

With English Notes.

Second Edition. 172 pp. (1858). Fcap. 8vo. 4s. 6d.

"This School edition of Sallust is precisely what the School edition of a Latin author ought to be. No useless words are spent in it, and no words that could be of use are spared. The text has been carefully collated with the best editions. With the work is given a full current of extremely well-selected annotations."—THE EXAMINER.

The "CATILINA" and "JUGURTHA" may be had separately, price 2s. 6d. each, bound in cloth.

By B. DRAKE, M.A.

Late Fellow of King's Coll. Cambridge.

1. Demosthenes on the Crown.

With English Notes.

Second Edition. To which is prefixed ÆSCHINES AGAINST CTESIPHON. With English Notes.

287 pp. (1860). Fcap. 8vo. cloth. 5s.

The first edition of the late Mr. Drake's edition of Demosthenes & Corona having met with considerable acceptance in various Schools, and a new edition being called for, the Oration of Æschines against Ctesiphon, in accordance with the wishes of many teachers, has been appended with useful notes by a competent scholar.

2. Æschyli Eumenides.

With English Verse Translation, Copious Introduction, and Notes.

8vo. 144. pp. (1853).

"Mr. Drake's ability as a critical Scholar is known and admitted. In the edition of the Eumenides before us we meet with him also in the capacity of a Poet and Historical Essayist. The translation is flowing and melodious, elegant and scholarlike. The Greek Text is well printed: the notes are clear and useful."—GUARDIAN.

By P. FROST, Jun., M.A.

Late Fellow of St. John's Coll. Cambridge.

Thucydides. Book VI.

With English Notes, Map and Index.

8vo. cloth. 7s. 6d.

It has been attempted in this work to facilitate the attainment of accuracy in translation. With this end in view the text has been treated grammatically.

By J. WRIGHT, M.A.

Head Master of Sutton Coldfield School.

Help to Latin Grammar.

With Easy Exercises, both English and Latin, Questions and Vocabulary.

175 pp. (1855). Crown 8vo. cloth. 4s. 6d.

"This book aims at helping the learner to overstep the threshold difficulties of the Latin Grammar; and never was there a better aid offered alike to teacher and scholar in that arduous pass. The style is at once familiar and strikingly simple and lucid; and the explanations precisely hit the difficulties, and thoroughly explain them. It will also much facilitate the acquirement of English Grammar."—ENGLISH JOURNAL OF EDUCATION.

2. Hellenica.

A FIRST GREEK READING BOOK.

From Diodorus and Thucydides. With Vocabulary.

Second Edition. 150 pp. (1851). Fcap. 8vo. cloth. 3s. 6d.

In the last twenty chapters of this volume, Thucydides sketches the rise and progress of the Athenian Empire in so clear a style and in such simple language, that the author doubts whether any easier or more instructive passages can be selected for the use of the pupil who is commencing Greek.

By J. WRIGHT, M.A.

Head Master of Sutton Coldfield School.

3. The Seven Kings of Rome.

A First Latin Reading Book, abridged from Livy, by the omission of difficult passages, with Notes and Index.

Second Edition. 138 pp. (1857). Fcap. 8vo. cloth. 3s.

This work is intended to supply the pupil with an easy Construing-book, which may, at the same time, be made the vehicle for instructing him in the rules of grammar and principles of composition. These branches of the study of Latin seem to the author to have hitherto been kept too much apart. Boys have construed their Delectus, or Eutropius, or Nepos, and have gone elsewhere for their grammatical exercises. Nor can this be wondered at. An educated man must feel positively ashamed of taking his pupils away from our good English authors, and setting before him instead a Delectus or Eutropius. He therefore skims over them as lightly, and escapes from them as quickly as possible, and has recourse for his composition lesson to one of the many Exercise-books which swarm from our educational presses. To remedy these evils this book has been published. Here Livy tells his own pleasant stories in his own pleasant words. What is omitted, is that which no one can wish a beginner to learn, and which may be better learnt elsewhere. Let Livy be the master to teach a boy Latin, not some English collector of sentences, and he will not be found a dull one.

4. Vocabulary and Exercises on "The Seven Kings of Rome."

94 pp. (1857). Crown, 8vo. cloth. 2s. 6d.

The Vocabulary is published apart from the Text in order to suit the views of those who prefer their pupils to consult a general dictionary, but it may also be had bound together with the "SEVEN KINGS OF ROME," if preferred. As the aim of the Text is to teach the elements of grammar, so the Exercises are intended to test the Pupil's knowledge of grammar. Indeed there is hardly an ordinary Latin construction which is not illustrated in the text, explained in the notes, and proved in the Exercises.

By EDWARD THRING, M.A.

Head Master of Uppingham School.

1. Elements of Grammar Taught in English.

With Questions.

Third Edition. 136 pp. (1860). Demy 18mo. 2s.

2. The Child's English Grammar.

New Edition. 86 pp. (1859). Demy 18mo. 1s.

The Author's effort in these two books has been to point out the broad, beaten, every-day path, carefully avoiding digressions into the byways and eccentricities of language. This Work took its rise from questionings in National Schools, and the whole of the first part is merely the writing out in order the answers to questions which have been used already with success. The study of Grammar in English has been much neglected, nay by some put on one side as an impossibility. There was perhaps much ground for this opinion, in the medley of arbitrary rules thrown before the student, which applied indeed to a certain number of instances, but would not work at all in many others, as must always be the case when principles are not put forward in a language full of ambiguities. The present work does not, therefore, pretend to be a compendium of idioms, or a philological treatise, but a Grammar. Or in other words, its intention is to teach the learner how to speak and write correctly, and to understand and explain the speech and writings of others. Its success, not only in National Schools, from practical work in which it took its rise, but also in classical schools, is full of encouragement.

3. School Songs.

A COLLECTION OF SONGS FOR SCHOOLS.

WITH THE MUSIC ARRANGED FOR FOUR VOICES.

Edited by Rev. E. THRING and H. RICCIUS.

Music Size. 7s. 6d.

By EDWARD THRING, M.A.

Head Master of Uppingham School.

4. **A First Latin Construing Book.**

104 pp. (1855). Fcap. 8vo. 2s. 6d.

This Construing Book is drawn up on the same sort of graduated scale as the Author's *English Grammar*. Passages out of the best Latin Poets are gradually built up into their perfect shape. The few words altered, or inserted as the passages go on, are printed in Italics. It is hoped by this plan that the learner, whilst acquiring the rudiments of language, may store his mind with good poetry and a good vocabulary.

Religious Class Books.

1.

By C. J. VAUGHAN, D.D.

Head Master of Harrow School.

St. Paul's Epistle to the Romans.

The Greek Text with English Notes.

Second Edition. Crown 8vo. cloth. (1861). 5s.

By dedicating this work to his elder Pupils at Harrow, the Author hopes that he sufficiently indicates what is and what is not to be looked for in it. He desires to record his impression, derived from the experience of many years, that the Epistles of the New Testament, no less than the Gospels, are capable of furnishing useful and solid instruction to the highest classes of our Public Schools. If they are taught accurately, not controversially; positively, not negatively; authoritatively, yet not dogmatically; taught with close and constant reference to their literal meaning, to the connexion of their parts, to the sequence of their argument, as well as to their moral and spiritual instruction; they will interest, they will inform, they will elevate; they will inspire a reverence for Scripture never to be discarded, they will awaken a desire to drink more deeply of the Word of God, certain hereafter to be gratified and fulfilled.

2.

Notes for Lectures on Confirmation: With Suitable Prayers. By C. J. VAUGHAN, D.D. Fourth Edition. 72 pp. (1862). Fcp. 8vo. 1s. 6d.

This work, originally prepared for the use of Harrow School, is published in the belief that it may assist the labours of those who are engaged in preparing candidates for Confirmation, and who find it difficult to lay their hand upon any one book of suitable instruction at once sufficiently full to furnish a synopsis of the subject, and sufficiently elastic to give free scope to the individual judgment in the use of it.

3.

The Church Catechism Illustrated and Explained. By ARTHUR RAMSAY, M.A. 18mo. cloth.

4.

Hand-Book to Butler's Analogy. By C. A. SWAINSON, M.A. 55 pp. (1856). Crown 8vo. 1s. 6d.

5.

History of the Christian Church during the First Three Centuries, and the Reformation in England. By W. SIMPSON, M.A. Fourth Edition. Fcp. 8vo. cloth. 3s. 6d.

6.

Analysis of Paley's Evidences of Christianity. By CHARLES H. CROSSE, M.A. 115 pp. (1855). 18mo. 3s. 6d.

CAMBRIDGE MANUALS

FOR THEOLOGICAL STUDENTS.

UNIFORMLY PRINTED AND BOUND.

This Series of THEOLOGICAL MANUALS has been published with the aim of supplying Books concise, comprehensive, and accurate; convenient for the Student, and yet interesting to the general reader.

1. **History of the Christian Church during the Middle Ages.** By ARCHDEACON HARDWICK. Second Edition. 482 pp. (1861). With Maps. Crown 8vo. cloth. 70s. 6d.

This Volume claims to be regarded as an integral and independent treatise on the Medieval Church. The History commences with the time of Gregory the Great, because it is admitted on all hands that his pontificate became a turning-point, not only in the fortunes of the Western tribes and nations, but of Christendom at large. A kindred reason has suggested the propriety of publishing at the year 1520,—the year when Luther, having been extruded from those Churches that adhered to the Communion of the Pope, established a provisional form of government and opened a fresh era in the history of Europe.

2. **History of the Christian Church during the Reformation.** By ARCHDEACON HARDWICK. 459 pp. (1856). Crown 8vo. cloth. 10s. 6d.

This Work forms a Sequel to the Author's Book on The Middle Ages. The Author's wish has been to give the reader a trustworthy version of those stirring incidents which mark the Reformation period.

3. **History of the Book of Common Prayer.** With a Rationale of its Offices. By FRANCIS PROCTER, M.A. Fifth Edition. 464 pp. (1860). Crown 8vo. cloth. 10s. 6d.

The Subject of this Book has been already treated by numerous writers of distinction. When the present series of Manuals was projected, it did not appear that any one of the existing volumes taken singly was avail-

able for the desired object. In the course of the last twenty years the whole question of liturgical knowledge has been reopened with great learning and accurate research, and it is mainly with the view of epitomizing their extensive publications, and correcting by their help the errors and misconceptions which had obtained currency, that the present volume has been put together.

4. **History of the Canon of the New Testament during the First Four Centuries.** By BROOKE FOSS WESTCOTT, M.A. 594 pp. (1855). Crown 8vo. cloth. 12s. 6d.

The Author has endeavoured to connect the history of the New Testament Canon with the growth and consolidation of the Church, and to point out the relation existing between the amount of evidence for the authenticity of its component parts and the whole mass of Christian literature. Such a method of inquiry will convey both the truest notion of the connexion of the written Word with the living Body of Christ, and the surest conviction of its divine authority.

5. **Introduction to the Study of the GOSPELS.** By BROOKE FOSS WESTCOTT, M.A. 458 pp. (1860). Crown 8vo. cloth. 10s. 6d.

The title of this book will explain the chief aim which the Author had in view. It is intended to be an Introduction to the Study of the Gospels. The Author has therefore confined himself in many cases to the mere indication of lines of thought and inquiry from the conviction that truth is felt to be more precious in proportion as it is opened to us by our own work. In a subject which involves so vast a literature much must have been overlooked; but the Author has made it a point at least to study the researches of the great writers, and consciously to neglect none.

FORTHCOMING BOOKS.

An Elementary Treatise on the Planetary Theory.

By C. H. H. CHEYNE, B.A., Scholar of St. John's College. Crow
8vo. cloth. 6s. 6d. [Just ready.]

2.

A Treatise on Geometry of Three Dimensions.

By PERCIVAL FROST, M.A., St. John's College, and JOSEPH
WOLSTENHOLME, M.A., Christ's College, Cambridge. 8vo. 18s.

*The First Portion has been issued for the convenience of Cambridge
Students: the concluding portion will appear before Christmas, 1862.*

3.

An Elementary Treatise on Natural Philosophy.

By WILLIAM THOMSON, LL.D., F.R.S., late Fellow of St. Peter's
College, Cambridge, Professor of Natural Philosophy in the University
of Glasgow; and PETER GUTHRIE TAIT, M.A., late Fellow of
St. Peter's College, Cambridge, Professor of Natural Philosophy in the
University of Edinburgh. With numerous Illustrations. [Preparing.]

4.

An Elementary Treatise on Quaternions.

With numerous Examples. By P. G. TAIT, M.A., Professor of
Natural Philosophy in the University of Edinburgh. [Preparing.]

5.

First Book of Algebra. For Schools.

By J. C. W. ELLIS, M.A., and P. M. CLARKE, M.A., Sidney Sussex
College, Cambridge. [Preparing.]

6.

Aristotelis de Rhetorica.

With Notes and Introduction. By E. M. COPE, M.A., Fellow and
Assistant-Tutor of Trinity College, Cambridge.

7.

The New Testament in the Original Greek.

Text revised by B. E. WESTCOTT, M.A., and F. J. HORT, M.A.,
formerly Fellows of Trinity College.

MACMILLAN AND CO.'S
CAMBRIDGE SCHOOL CLASS BOOKS.

I.

Euclid for Colleges and Schools.

By I. TODHUNTER, M.A., F.R.S., Fellow and Principal Mathematical Lecturer of St. John's College, Cambridge. 18mo. 3s. 6d.

II.

An Elementary Latin Grammar.

By H. J. ROBY, M.A., Under Master of Dulwich College Upper School, late Fellow and Classical Lecturer of St. John's College, Cambridge. 18mo. 2s. 6d.

III.

An Elementary History of the Book of Common Prayer.

By FRANCIS PROCTER, M.A., Vicar of Witton, Norfolk, late Fellow of St. Catharine's College, Cambridge. 18mo. 2s. 6d.

These volumes, forming the first portion of Macmillan and Co.'s ELEMENTARY SCHOOL CLASS BOOKS, are handsomely printed and bound. All the volumes of the series will be published at a low price to ensure an extensive sale in the Schools of the United Kingdom and the Colonies.

The following will shortly be published:

The School Class Book of Arithmetic.

By BARNARD SMITH, M.A., late Fellow of St. Peter's College, Cambridge.

Algebra for Beginners.

By I. TODHUNTER, M.A., F.R.S.

The Bible Word-Book.

A Glossary of old English Bible Words with Illustrations.

By J. EASTWOOD, M.A., St. John's College, Cambridge, and Incumbent of Hope, near Sheffield, and W. ALDIS WRIGHT, M.A., Trinity College, Cambridge.

* * * Other Volumes will be announced in due course.

Printed by Jonathan Palmer, Sidney Street, Cambridge.

